To the Hon. G. W. Ross
with the Author's Compts.

FORMS, NECESSARY AND SUFFICIENT, OF THE ROOTS OF PURE UNI-SERIAL ABELIAN EQUATIONS.

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Forms, Necessary and Sufficient, of the Roots of Pure Uni-Serial Abelian Equations.

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Object of the Paper.

§1. An Abelian equation, that is, an irreducible equation in which one root is a rational function of another and of known quantities, may be called *uni-serial* when the roots form a single circulating series. If the equation, say $f(x) = 0$, be of the n^{th} degree, its roots, in the ordinary Abelian notation, are

$$x_1,\ \theta x_1,\ \theta^2 x_1,\ \ldots\ldots,\ \theta^{n-1} x_1. \tag{1}$$

§2. When the coefficients of θ are rational, in other words, when one root of the equation $f(x) = 0$ is a rational function of another, the equation is a *pure* Abelian. For instance, the irreducible cubic equation

$$x^3 + px + q = 0,$$

in which the coefficients p and q are such that $\sqrt{(-4p^3 - 27q^2)}$ is rational, is a pure Abelian, because, as is well known, one root of the cubic is a rational function of either of the others.

§3. The object of the following paper is to investigate the necessary and sufficient forms of the roots of pure uni-serial Abelian equations. First, a Criterion of pure uni-serial Abelianism is established (§12–§15). A deduction is then given of the necessary and sufficient forms of the roots of pure uni-serial Abelian equations of all prime degrees (§16–§26). Then the necessary and sufficient forms of the roots of the pure uni-serial Abelian quartic are obtained by two different methods (§27–§39). Then the necessary and sufficient forms of the roots of the pure uni-serial Abelian of a degree which is the continued product of any number of distinct prime numbers are found (§40–§46). Then the problem is solved for the pure uni-serial Abelian of a degree which is four times the continued product of any number of distinct odd

primes (§47–§57). Finally, from the relation between the solvable irreducible equation of prime degree n and the pure uni-serial Abelian equation of degree $n-1$, the necessary and sufficient forms of the roots of the irreducible solvable equation of prime degree n are shown to be determinable for all cases in which $n-1$ is either the continued product of a number of distinct primes, or four times the continued product of a number of distinct odd primes (§58–§64).

PRELIMINARY.

Corollary from a Law of Kronecker.

§4. It was proved by Kronecker that, n being any integer, the primitive n^{th} roots of unity are the roots of an irreducible equation, that is, of an irreducible equation with rational coefficients. We shall have occasion to make use of the following Corollary from this law: Let w and w' be two primitive n^{th} roots of unity, and let $F(w)$ be a rational function of w. Then, if $F(w)=0$, $F(w')=0$. For, by hypothesis,

$$F(w)=hw^s+h_1w^{s-1}+\text{ etc.}=0,$$

where h, h_1, etc., are rational. We assume s to be less than n, and h to be distinct from zero; therefore

$$h^{-1}\{F(w)\}=w^s+h^{-1}h_1w^{s-1}+\text{ etc.}=0.$$

Therefore w is a root of the equation $\phi(x)=x^s+h^{-1}h_1x^{s-1}+\text{ etc.}=0$. If $\psi(x)=0$ be the equation whose roots are the primitive n^{th} roots of unity, w is a root of the equation $\psi(x)=0$. Therefore the equations $\phi(x)=0$ and $\psi(x)=0$ have a root in common. But, by Kronecker's law, the equation $\psi(x)=0$ is irreducible. Therefore $\phi(x)$ is divisible by $\psi(x)$ without remainder. This implies that all the roots of the equation $\psi(x)=0$ are roots of the equation $\phi(x)=0$. Therefore $\phi(w')=0$. Therefore $F(w')=0$.

Principles established by Abel.

§5. Let $f(x)=0$ be a uni-serial Abelian equation of the n^{th} degree, and let its roots, in the order in which they circulate, be the terms in (1). It is known (see Serret's Cours d'Algèbre supérieure, Vol. II, page 500, third edition) that

$$x_1=R_0^{\frac{1}{n}}+R_1^{\frac{1}{n}}+R_2^{\frac{1}{n}}+\ldots\ldots+R_{n-1}^{\frac{1}{n}},$$

where R_1 is a rational function of the primitive n^{th} root of unity w and of the known quantities involved in the coefficients of θ; and, z being any integer, R_z is derived from R_1 by changing w into w^z. Putting

$$x_{s+1} = R_0^{\frac{1}{n}} + w^s R_1^{\frac{1}{n}} + w^{2s} R_2^{\frac{1}{n}} + \ldots . + w^{(n-1)s} R_{n-1}^{\frac{1}{n}}, \tag{2}$$

the n roots of the equation $f(x) = 0$ are obtained by giving s in x_{s+1} successively the values $0, 1, 2, \ldots ., n-1$. Therefore $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation; consequently, $R_0^{\frac{1}{n}}$ is rational. An equation of the type

$$(R_z R_1^{-z})^{\frac{1}{n}} = F(w) \tag{3}$$

subsists for every integral value of z, $F(w)$ being a rational function of w and of the known quantities involved in the coefficients of θ. As w may be any one of the primitive n^{th} roots of unity, if the general primitive n^{th} root of unity be w^e, we may suppose w in R_1 to be changed into w^e. The n roots of the equation $f(x) = 0$ will then be obtained by giving t, in the expression

$$R_0^{\frac{1}{n}} + w^t R_e^{\frac{1}{n}} + w^{2t} R_{2e}^{\frac{1}{n}} + \text{etc.} \tag{4}$$

successively the values $0, 1, 2, \ldots ., n-1$. Abel's investigation shows that the form of the function $F(w)$ in (3) is independent of the particular primitive n^{th} root of unity denoted by w. Hence the change of w into w^e causes equation (3) to become

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = F(w^e), \tag{5}$$

the symbol F having the same meaning for every value of e.

Fundamental Element of the Root.

§6. Because R_0, R_2, etc., are derived from R_1 by changing w into w^0, w^2, etc., the root x_1 can be constructed when R_1 is given. We may therefore call R_1 the fundamental element of the root. Examples of the way in which the root is constructed from its fundamental element will present themselves in the course of the paper.

A Certain Rational Function of the Primitive n^{th} Root of Unity, n being an Odd Prime Number.

§7. Taking n an odd prime number, there is a certain rational function of the primitive n^{th} root of unity w, of which we shall have occasion to make

frequent use. It will be convenient to describe it here, and to point out two of its properties. Let

$$w,\ w^{\lambda},\ w^{\lambda^2},\ \ldots\ldots,\ w^{\lambda^{n-2}}, \tag{6}$$

be a cycle containing all the primitive n^{th} roots of unity. The number λ may be assumed to be less than n. With a view to convenience in printing, the indices of the powers of w in (6) may be written

$$1,\ \lambda,\ \alpha,\ \beta,\ \ldots\ldots,\ \delta,\ \varepsilon,\ \theta; \tag{7}$$

that is to say, $\alpha = \lambda^2$, $\beta = \lambda^3$, and so on. Take P_1 a rational function of w, and, z being any integer, let P_z be what P_1 becomes when w is changed into w^z. Then the function to which we desire to call attention is

$$P_1^{\theta} P_{\lambda}^{\varepsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\delta}^{\alpha} P_{\varepsilon}^{\lambda} P_{\theta}. \tag{8}$$

The *subscripts* of the factors of the expression (8) are the terms in (7), while the *indices* are the terms in (7) in reverse order. The expression (8) may be denoted by the symbol ϕ_1. From ϕ_1, as expressed in (8), derive ϕ_z by changing w into w^z, z being any integer. Then

$$\left.\begin{array}{l} \phi_1 = P_1^{\theta} P_{\lambda}^{\varepsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\delta}^{\alpha} P_{\varepsilon}^{\lambda} P_{\theta} \\ \phi_{\lambda} = P_1 P_{\lambda}^{\theta} P_{\alpha}^{\varepsilon} \ldots\ldots \qquad P_{\varepsilon}^{\alpha} P_{\theta}^{\lambda} \\ \phi_{\alpha} = P_1^{\lambda} P_{\lambda} P_{\alpha}^{\theta} \ldots\ldots \qquad\quad P_{\theta}^{\alpha} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \phi_{\theta} = P_1^{\varepsilon} P_{\lambda}^{\delta} \qquad \ldots\ldots P_{\delta}^{\lambda} P_{\varepsilon} P_{\theta}^{\theta} \end{array}\right\} \tag{9}$$

The second of these equations is derived from the first by changing w into w^{λ}. This, since $\alpha = \lambda^2$ and $\beta = \lambda^3$, and so on, causes w^{λ} to become w^{α}, and w^{α} to become w^{β}, and so on. Hence it causes P_{λ} to become P_{α}, P_{α} to become P_{β}, and so on. Thus the second of equations (9) is obtained. The rest are obtained in a similar manner.

§8. One property which the function ϕ_1 possesses is that $\phi_0^{\frac{1}{n}}$ has a rational value. For

$$\phi_0 = P_0^{\theta} P_0^{\varepsilon} \ldots\ldots P_0^{\lambda} P_0 = P_0^{t},$$

where

$$t = 1 + \lambda + \lambda^2 + \ldots\ldots + \lambda^{n-2} = \frac{\lambda^{n-1} - 1}{\lambda - 1}.$$

Because (6) is a cycle of primitive n^{th} roots of unity, $\lambda^{n-1} - 1$ is a multiple of n. And, since λ is less than n, $\lambda - 1$ is not a multiple of n; therefore t is a multiple of n. Put $t = mn$; then

$$\phi_0 = (P_0^m)^n;$$

consequently, one of the values of $\phi_0^{\frac{1}{n}}$ is the rational quantity P_0^m.

§ 9. A second property of the function ϕ_1 is that an equation of the type

$$(\phi_z\phi_1^{-z})^{\frac{1}{n}} = F(w) \tag{10}$$

subsists for every integral value of z, $F(w)$ being a rational function of w. For, taking $z = \lambda$,

$$\phi_1^\lambda = P_1^{\lambda\theta} P_\lambda^{\lambda\epsilon} P_\alpha^{\lambda\delta} \dots P_\delta^{\lambda\alpha} P_\epsilon^{\lambda^2} P_\theta^\lambda.$$

But $\lambda^2 = \alpha$, $\alpha\lambda = \beta$, ..., $\lambda\epsilon = \theta$. And $\lambda\theta = \lambda^{n-1}$. Since $\lambda^{n-1} - 1$ is a multiple of n, put $\lambda^{n-1} - 1 = cn$. Then

$$\phi_1^\lambda = P_1^{cn}(P_1 P_\lambda^\theta \dots P_\theta^\lambda).$$

Comparing this with the second of equations (9),

$$\phi_\lambda\phi_1^{-\lambda} = P_1^{-cn}.$$

Therefore

$$(\phi_\lambda\phi_1^{-\lambda})^{\frac{1}{n}} = w' P_1^{-c}, \tag{11}$$

w' being an n^{th} root of unity. In like manner, from the second and third of equations (9),

$$\phi_\alpha\phi_\lambda^{-\lambda} = P_\lambda^{-cn}.$$

Substitute here the value of ϕ_λ in (11). Then $\phi_\alpha\phi_1^{-\alpha} = (P_\lambda^{-c} P_1^{-\lambda c})^n$. Therefore

$$(\phi_\alpha\phi_1^{-\alpha})^{\frac{1}{n}} = w''(P_\lambda^{-c} P_1^{-\lambda c}), \tag{12}$$

w'' being an n^{th} root of unity. The equations (11) and (12) are of the type (10). Therefore an equation of the type (10) subsists when z is equal either to λ or to α. In the same way we can go on to show that an equation of the type (10) subsists when z is equal to any of the terms in (7). Should $z = 0$, $\phi_z\phi_1^{-z} = \phi_0$. Therefore, by § 8, $\phi_z\phi_1^{-z} = P_0^{mn}$. Hence in this case also $(\phi_z\phi_1^{-z})^{\frac{1}{n}}$ is a rational function of w. Therefore, whether z be zero or one of the terms in the series (7), an equation of the type (10) subsists. This implies that an equation of the type (10) subsists for every integral value of z.

Criterion of Pure Uni-Serial Abelianism.

The Criterion Stated.

§ 10. A Criterion of pure uni-serial Abelianism may now be given. Let R_1 be a rational function of the primitive n^{th} root of unity w, and, z being any integer, let R_z be derived from R_1 by changing w into w^z. Then, if $R_0^{\frac{1}{n}}$ is rational, and if the terms $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., are such that an equation of the type

(3) subsists for every integral value of z, an equation (5), in which the symbol F has the same meaning as in (3), at the same time subsisting for every value of e prime to n, the n values of x_{s+1} in (2), obtained by giving s successively the values $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation, provided always that the equation of the n^{th} degree, of which they can be shown to be the roots, is irreducible.

Proof of the Criterion.

§11. Here we assume that the conditions specified in §10 are satisfied, and we have to show that the n values of x_{s+1} in (2), obtained by putting s successively equal to $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation.

§12. We will first prove that the n values of the expression (4) obtained by giving t successively the n values $0, 1, 2, \ldots, n-1$, are the same, the order of the terms not being considered, as the n values of x_{s+1} in (2) obtained by giving s successively the values $0, 1, 2, \ldots, n-1$.

Because w^e is a primitive n^{th} root of unity, all the n^{th} roots of unity distinct from unity are contained in the series

$$w^e,\ w^{2e},\ w^{3e}, \ldots, w^{(n-1)e}.$$

Therefore the two series

$$R_1,\ R_2,\ R_3, \ldots, R_{n-1},$$
$$R_e,\ R_{2e},\ R_{3e}, \ldots, R_{(n-1)e},$$

are identical with one another, the order of the terms not being considered. Therefore, also, the two series

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ R_3^{\frac{1}{n}}, \ldots, R_{n-1}^{\frac{1}{n}},$$
$$R_e^{\frac{1}{n}},\ R_{2e}^{\frac{1}{n}},\ R_{3e}^{\frac{1}{n}}, \ldots, R_{(n-1)e}^{\frac{1}{n}},$$

are identical with one another, the order of the terms not being considered, it being understood that $R_e^{\frac{1}{n}}$, $R_{2e}^{\frac{1}{n}}$, etc., are the same n^{th} roots of R_e, R_{2e}, etc., or of R_1, R_2, etc., that are taken in the series $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc. Let the expression (4) be called x'_{t+1}. The separate members of the expression x_{s+1} are

$$R_0^{\frac{1}{n}},\ w^s R_1^{\frac{1}{n}},\ w^{2s} R_2^{\frac{1}{n}}, \text{ etc.} \tag{13}$$

Taking s with a definite value, let

$$es = bn + c,$$

where b and c are whole numbers, and c is less than n. Then, putting $t = c$, the separate members of the expression x'_{c+1} are

$$R_0^{\frac{1}{n}},\ w^c R_e^{\frac{1}{n}},\ w^{2c} R_{2e}^{\frac{1}{n}},\ \text{etc.} \tag{14}$$

Because $es = bn + c$, $w^c = w^{es}$. Therefore $w^c R_e^{\frac{1}{n}} = w^{es} R_e^{\frac{1}{n}}$; that is, the second term in (14) is equal to the $(e+1)^{\text{th}}$ term in (13). Again, if $2e = dn + v$, where d and v are whole numbers, and v is less than n, $R_{2e}^{\frac{1}{n}} = R_v^{\frac{1}{n}}$. Also, because $es = bn + c$, $w^{2c} = w^{2es}$. Therefore $w^{2c} R_{2e}^{\frac{1}{n}} = w^{2es} R_v^{\frac{1}{n}} = w^{vs} R_v^{\frac{1}{n}}$; that is, the third term in (14) is equal to the $(v+1)^{\text{th}}$ term in (13); and so on. Hence $x'_{c+1} = x_{s+1}$. Let now s and σ be two distinct values of s, both less than n; and let

$$x'_{c+1} = x_{s+1},\ \text{and}\ x'_{z+1} = x_{\sigma+1}.$$

By what has been proved, the numbers c and z are determined by the equations

$$es = bn + c,\ e\sigma = \beta n + z,$$

bn and βn being multiples of n. But, since s and σ are different, and e is prime to n, c and z must be different. Hence, as x_{s+1} runs through its n values, x_1, x_2, etc., x'_{t+1} must run through its n values, severally equal, in some order, to those of x_{s+1}.

§13. From (5),

$$\begin{aligned} R_{2e}^{\frac{1}{n}} &= A_e R_e^{\frac{2}{n}}, \\ R_{3e}^{\frac{1}{n}} &= B_e R_e^{\frac{3}{n}}, \\ &\ldots\ldots\ldots\ldots \\ R_{(n-1)e}^{\frac{1}{n}} &= C_e R_e^{\frac{n-1}{n}}, \end{aligned}$$

where A_e, B_e, etc., are rational functions of w^e. These values of $R_{2e}^{\frac{1}{n}}$, $R_{3e}^{\frac{1}{n}}$, etc., substituted in (4), cause that expression to become

$$R_0^{\frac{1}{n}} + w^t R_e^{\frac{1}{n}} + w^{2t} A_e R_e^{\frac{2}{n}} + w^{3t} B_e R_e^{\frac{3}{n}} + \text{etc.} \tag{15}$$

Let the n values of the expression (15), obtained by putting t successively equal to $0, 1, 2, \ldots\ldots, n-1$, be

$$r_1,\ r_2,\ \ldots\ldots,\ r_n. \tag{16}$$

Then, v being a whole number,

$$\begin{aligned} r_1^v &= a_e + b_e R_e^{\frac{1}{n}} + c_e R_e^{\frac{2}{n}} + \ldots\ldots + d_e R_e^{\frac{n-1}{n}}, \\ r_2^v &= a_e + w b_e R_e^{\frac{1}{n}} + w^2 c_e R_e^{\frac{2}{n}} + \ldots\ldots + w^{(n-1)} d_e R_e^{\frac{n-1}{n}}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ r_n^v &= a_e + w^{-1} b_e R_e^{\frac{1}{n}} + w^{-2} c_e R_e^{\frac{2}{n}} + \ldots\ldots + w d_e R_e^{\frac{n-1}{n}}, \end{aligned}$$

where a_e, b_e, etc., are rational functions of w^e. Therefore, if S_v be the sum of the v^{th} powers of the terms in (16), $S_v = na_e$. Because a_e is a rational function of w^e, we may put

$$na_e = g + hw^e + kw^{2e} + \dots + lw^{(n-1)e}, \text{ where } g,\ h, \text{ etc., are rational.}$$

But, by § 12, the n values of the expression (15), obtained by giving t successively the values $0, 1, 2, \dots, n-1$, are the same whatever value, making w^e a primitive n^{th} root of unity, be given to e. We may therefore substitute for w^e, in the expression for na_e or S_v, any one of the primitive n^{th} roots of unity

$$w,\ w^c,\ w^d,\ \dots,\ w^z. \tag{17}$$

Therefore

$$\begin{aligned} S_v &= g + hw + kw^2 + \text{etc.} \\ &= g + hw^c + kw^{2c} + \text{etc.} \\ &\dots\dots\dots\dots\dots\dots \\ &= g + hw^z + kw^{2z} + \text{etc.} \end{aligned}$$

Therefore

$$mS_v = mg + h(w + w^c + \text{etc.}) + k(w^2 + w^{2c} + \text{etc.}) + \text{etc.},$$

m being the number of the terms in the series (17). Consequently S_v is a rational and symmetrical function of the primitive n^{th} roots of unity. Hence, by Kronecker's law, referred to in § 4, S_v is rational. This implies that the n terms in (16), which have been shown to be identical with the n values of x_{s+1} in (2) obtained by giving s successively the values $0, 1, 2, \dots, n-1$, are the roots of an equation of the n^{th} degree; that is, of an equation of the n^{th} degree with rational coefficients. Let this equation be $f(x) = 0$.

§ 14. In accordance with the proviso in § 10, let the equation $f(x) = 0$ be irreducible. It is then a pure Abelian. For, taking r_1, r_2, etc., as in § 13,

$$\left.\begin{aligned} r_1 &= R_0^{\frac{1}{n}} + R_e^{\frac{1}{n}} + A_eR_e^{\frac{2}{n}} + \dots + C_eR_e^{\frac{n-1}{n}} \\ r_1^2 &= D_e + F_eR_e^{\frac{1}{n}} + G_eR_e^{\frac{2}{n}} + \dots + H_eR_e^{\frac{n-1}{n}} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ r_1^{n-1} &= K_e + L_eR_e^{\frac{1}{n}} + M_eR_e^{\frac{2}{n}} + \dots + Q_eR_e^{\frac{n-1}{n}} \end{aligned}\right\} \tag{18}$$

where A_e, D_e, F_e, etc., are rational functions of w^e. Multiply the first of equations (18) by h_e, the second by k_e, and so on, the last being multiplied by l_e; then, by addition,

$$\begin{aligned} h_er_1 + k_er_1^2 + \dots + l_er_1^{n-1} = &\left(h_eR_0^{\frac{1}{n}} + k_eD_e + \dots + l_eK_e\right) \\ &+ (h_e + k_eF_e + \dots + l_eL_e)R_e^{\frac{1}{n}} \\ &+ \dots\dots\dots\dots\dots\dots \\ &+ (h_eC_e + k_eH_e + \dots + l_eQ_e)R_e^{\frac{n-1}{n}}. \end{aligned}$$

Let the $n-1$ quantities, h_e, k_e, etc., be determined by the $n-1$ equations

$$h_e \quad + k_e F_e + \ldots\ldots + l_e L_e = w^e,$$
$$h_e A_e + k_e G_e + \ldots\ldots + l_e M_e = w^{2e} A_e,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$h_e C_e + k_e H_e + \ldots\ldots + l^e Q_e = w^{(n-1)e} C_e.$$

Then
$$h_e r_1 + k_e r_1^2 + \text{etc.} = \left(h_e R_0^{\frac{1}{n}} + k_e D_e + \ldots\ldots + l_e K_e\right)$$
$$+ w^e R_e^{\frac{1}{n}} + w^{2e} A_e R_e^{\frac{2}{n}} + \ldots\ldots + w^{(n-1)e} C_e R_e^{\frac{n-1}{n}};$$

or, putting $R_{2e}^{\frac{1}{n}}$ for $A_e R_e^{\frac{2}{n}}$, and so on,

$$h_e r_1 + k_e r_1^2 + \text{etc.} = \left(h_e R_0^{\frac{1}{n}} + k_e D_e + \ldots\ldots + l_e K_e\right)$$
$$+ w^e R_e^{\frac{1}{n}} + w^{2e} R_{2e}^{\frac{1}{n}} + \ldots\ldots + w^{(n-1)e} R_{(n-1)e}^{\frac{1}{n}}. \tag{19}$$

By § 12, $x'_{c+1} = x_{s+1}$, where $es = bn + c$. When $s = 0$, $c = 0$, and when $s = 1$, $c = e$; therefore

$$x_1 = x'_1 = R_0^{\frac{1}{n}} + R_e^{\frac{1}{n}} + R_{2e}^{\frac{1}{n}} + \text{etc.} = r_1,$$

and
$$x_2 = x'_{e+1} = R_0^{\frac{1}{n}} + w^e R_e^{\frac{1}{n}} + w^{2e} R_{2e}^{\frac{1}{n}} + \text{etc.}$$

Therefore (19) may be written

$$h_e x_1 + k_e x_1^2 + \text{etc.} = \left(h_e R_0^{\frac{1}{n}} + \text{etc.}\right) - R_0^{\frac{1}{n}} + x_2.$$

But e may be any number that makes w^e a primitive n^{th} root of unity, and (17) is the series of the primitive n^{th} roots of unity. Therefore

$$x_2 = \left\{R_0^{\frac{1}{n}} - \left(h_1 R_0^{\frac{1}{n}} + \text{etc.}\right)\right\} + h_1 x_1 + k_1 x_1^2 + \ldots\ldots + l_1 x_1^{n-1}$$
$$= \left\{R_0^{\frac{1}{n}} - \left(h_c R_0^{\frac{1}{n}} + \text{etc.}\right)\right\} + h_c x_1 + k_c x_1^2 + \ldots\ldots + l_c x_1^{n-1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= \left\{R_0^{\frac{1}{n}} - \left(h_s R_0^{\frac{1}{n}} + \text{etc.}\right)\right\} + h_s x_1 + k_s x_1^2 + \ldots\ldots + l_s x_1^{n-1},$$

where h_1, h_c, etc., are what h_e becomes when w^e is changed into w, w^c, etc., and k_1, k_c, etc., are what k_e becomes when w^e is changed into w, w^c, etc., and so on. Therefore, by addition, m being the number of the primitive n^{th} roots of unity,

$$mx_2 = p + qx_1 + tx_1^2 + \ldots\ldots + vx_1^{n-1},$$

where p, q, etc., are rational and symmetrical functions of the primitive n^{th} roots of unity, and therefore are rational. Hence x_2 is a rational function of x_1. Therefore the equation $f(x) = 0$ is a pure Abelian.

§ 15. It is also uni-serial. For, by what has been proved,

$$x_2 = \theta x_1,$$

θx_1 denoting a rational function of x_1. But, from the form of x_{s+1} in (2), since $R_2^{\frac{1}{n}} = A_1 R_1^{\frac{2}{n}}$, and $R_3^{\frac{1}{n}} = B_1 R_1^{\frac{3}{n}}$, and so on, we pass from x_1 to x_2 by simply changing $R_1^{\frac{1}{n}}$ into $wR_1^{\frac{1}{n}}$. The same change transforms x_2 into x_3. Therefore

$$x_3 = \theta x_2 = \theta^2 x_1.$$

In like manner $x_4 = \theta^3 x_1$, and so on, till ultimately $\theta^n x_1 = x_1$. Thus all the roots of the equation $f(x) = 0$ are comprised in the series

$$x_1,\ \theta x_1,\ \theta^2 x_1,\ \ldots\ldots,\ \theta^{n-1} x_1.$$

PURE ABELIAN EQUATIONS OF ODD PRIME DEGREES.

Fundamental Element of the Root; the Root Constructed from its Fundamental Element.

§ 16. We confine ourselves to pure Abelians of *odd* prime degrees, because the irreducible quadratic is always a pure Abelian. Let n be an odd prime number, and let the primitive n^{th} roots of unity be the terms w, w^{λ}, w^{λ^2}, etc., forming the series (6). Take ϕ_1 as in the first of equations (9); then, if R_1 be the fundamental element (see § 6) of the root of a pure Abelian equation $f(x) = 0$ of the n^{th} degree, it will be found that

$$R_1 = A_1^n \phi_1, \tag{20}$$

A_1 being a rational function of w.

§ 17. From R_1, as expressed in (20), derive R_0, R_2, etc., by changing w into w^0, w^2. etc. By § 5, the root of the equation $f(x) = 0$ is

$$R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots\ldots + R_{n-1}^{\frac{1}{n}}. \tag{21}$$

To construct the root, we have to determine the particular n^{th} roots of R_0, R_1, etc., that are to be taken together in (21). When w is changed into w^z, let A_1 become A_z, as ϕ_1 becomes ϕ_z. Then

$$R_z = A_z^n \phi_z.$$

Therefore

$$R_z^{\frac{1}{n}} = w' A_z \phi_z^{\frac{1}{n}}, \tag{22}$$

w' being an n^{th} root of unity. In proceeding to make $R_z^{\frac{1}{n}}$ definite, we may first make $\phi_z^{\frac{1}{n}}$ definite. By (9),

$$\phi_1^{\frac{1}{n}} = w^a \left(P_1^{\theta} P_{\lambda}^{\epsilon} P_{\alpha}^{\delta} \ldots\ldots P_{\theta}\right)^{\frac{1}{n}},$$

w^a being an n^{th} root of unity. Let

$$P_1^{\frac{1}{n}},\ P_\lambda^{\frac{1}{n}},\ P_a^{\frac{1}{n}},\ \ldots\ldots,\ P_\theta^{\frac{1}{n}}, \tag{23}$$

be determinate; then, by taking w^a with the value unity, we get $\phi_1^{\frac{1}{n}}$ with the determinate value

$$\phi_1^{\frac{1}{n}} = (P_1^\theta P_\lambda^\epsilon P_a^\delta \ldots\ldots P_\theta)^{\frac{1}{n}}.$$

Let us now consider $\phi_\lambda^{\frac{1}{n}}$. By (9), w^c being an n^{th} root of unity,

$$\phi_\lambda^{\frac{1}{n}} = w^c (P_1 P_\lambda^\theta P_a^\epsilon \ldots\ldots P_\theta^\lambda)^{\frac{1}{n}}.$$

Understanding that $P_1^{\frac{1}{n}}$, $P_\lambda^{\frac{1}{n}}$, etc., on the right-hand side of this equation are the same quantities that appear in (23), they have already been made definite. We can then make $\phi_\lambda^{\frac{1}{n}}$ definite by taking w^c with the value unity. Generally, if z be any number in the series $1, 2, \ldots\ldots, n-1$,

$$\phi_z^{\frac{1}{n}} = w^d (P_z^\theta P_{z\lambda}^\epsilon P_{za}^\delta \ldots\ldots P_{z\theta})^{\frac{1}{n}},$$

w^d being an n^{th} root of unity. Because z is prime to n, the $n-1$ terms $w^z, w^{z\lambda}, w^{za}, \ldots\ldots, w^{z\theta}$, are the same, in a certain order, with the terms $w, w^\lambda, w^a, \ldots\ldots, w^\theta$. Therefore the terms

$$P_z^{\frac{1}{n}},\ P_{z\lambda}^{\frac{1}{n}},\ P_{za}^{\frac{1}{n}},\ \ldots\ldots,\ P_{z\theta}^{\frac{1}{n}},$$

may be taken to be the same, in a certain order, with the terms in (23). They are accordingly determinate. We may then make $\phi_z^{\frac{1}{n}}$ definite by taking w^d with the value unity. Therefore, for every value of z in the series $1, 2, \ldots\ldots, n-1$,

$$\phi_z^{\frac{1}{n}} = (P_z^\theta P_{z\lambda}^\epsilon P_{za}^\delta \ldots\ldots P_{z\theta})^{\frac{1}{n}}. \tag{24}$$

Having thus determined $\phi_z^{\frac{1}{n}}$, we can make $R_z^{\frac{1}{n}}$ definite by taking w' in (22) equal to unity for every value of z in the series $1, 2, \ldots\ldots, n-1$; that is,

$$\left.\begin{array}{l} R_1^{\frac{1}{n}} = A_1 \phi_1^{\frac{1}{n}} \\ R_\lambda^{\frac{1}{n}} = A_\lambda \phi_\lambda^{\frac{1}{n}} \\ \ldots\ldots\ldots \\ R_\theta^{\frac{1}{n}} = A_\theta \phi_\theta^{\frac{1}{n}} \end{array}\right\} \tag{25}$$

As regards $R_0^{\frac{1}{n}}$, we have $R_0 = A_0^n \phi_0$. But, by § 8, $\phi_0 = P_0^{mn}$. Therefore $\phi_0^{\frac{1}{n}}$ has a rational value. Consequently $R_0^{\frac{1}{n}}$ has a rational value. In (21) substitute the rational value of $R_0^{\frac{1}{n}}$, and the values of $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, etc., given in (25), and the

root is constructed. In other words, the expression (21) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it can be shown to be the root, is irreducible.

Necessity of the above Forms.

§18. The root x_1 of the pure Abelian equation $f(x)=0$ of the n^{th} degree, n an odd prime, being assumed to be expressible as in (21), we have to show that its fundamental element R_1 has the form (20), and that $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, etc., are to be taken as in (25), while $R_0^{\frac{1}{n}}$ receives its rational value.

§19. By (3), z being any integer,

$$R_z^{\frac{1}{n}} = \{F(w)\}\, R_1^{\frac{z}{n}},$$

$F(w)$ being a rational function of w. And equation (5) subsists along with (3); that is, e being any whole number prime to n,

$$R_{ez}^{\frac{1}{n}} = \{F(w^e)\}\, R_e^{\frac{z}{n}}.$$

Give z here successively the values 1, λ, α, etc., these terms being the same as in the series (7). Then

$$\begin{aligned} R_e^{\frac{1}{n}} &= R_e^{\frac{1}{n}}, \\ R_{e\lambda}^{\frac{1}{n}} &= B_e R_e^{\frac{\lambda}{n}}, \\ R_{e\alpha}^{\frac{1}{n}} &= C_e R_e^{\frac{\alpha}{n}}, \\ &\dots\dots\dots \\ R_{e\theta}^{\frac{1}{n}} &= D_e R_e^{\frac{\theta}{n}}, \end{aligned}$$

B_e, C_e, etc., being rational functions of w^e. Therefore

$$(R_e^{\theta} R_{e\lambda}^{\varepsilon} R_{e\alpha}^{\delta} \dots R_{e\theta})^{\frac{1}{n}} = G_e R_e^{\frac{t}{n}},$$

where G_e is a rational function of w^e, and

$$t = \theta + \varepsilon\lambda + \delta\alpha + \dots + \theta.$$

From the nature of the series (7), $\theta = \lambda^{n-2}$, and $\varepsilon = \lambda^{n-3}$. Therefore $\varepsilon\lambda = \theta$. In like manner, each of the $n-1$ separate members of t is equal to θ. Therefore $t = (n-1)\,\theta$. Because (6) is a cycle of primitive n^{th} roots of unity, in other words, because λ is a prime root of n, and $\theta = \lambda^{n-2}$, θ is prime to n. And $n-1$ is necessarily prime to n. Therefore whole numbers h and k exist such that

$$ht = kn + 1.$$

Therefore
$$(R_e^\theta R_{e\lambda}^\epsilon \dots\dots R_{e\theta})^{\frac{h}{n}} = (G_e^h R_e^k) R_e^{\frac{1}{n}}.$$
For every integral value of z let $(R_{ez}^h)^{\frac{1}{n}}$ be written $P_{ez}^{\frac{1}{n}}$; then, putting A_e^{-1} for $G_e^h R_e^k$,
$$R_e^{\frac{1}{n}} = A_e (P_e^\theta P_{e\lambda}^\epsilon P_{e\alpha}^\delta \dots\dots P_{e\theta})^{\frac{1}{n}}. \tag{26}$$
Hence, by putting $e = 1$, and taking ϕ_1 as in (9),
$$R_1 = A_1^n \phi_1.$$
Thus the form of the fundamental element in (20) is established. Also, when $e = 1$,
$$R_1^{\frac{1}{n}} = A_1 (P_1^\theta P_\lambda^\epsilon P_\alpha^\delta \dots\dots P_\theta)^{\frac{1}{n}}.$$
Therefore, by (24), $R_1^{\frac{1}{n}} = A_1 \phi_1^{\frac{1}{n}}$. This is the first of equations (25). Since e may be any term prime to n, let $e = \lambda$. Then, from (26), because $\lambda^2 = \alpha$ and $\lambda\alpha = \beta$, and so on,
$$R_\lambda^{\frac{1}{n}} = A_\lambda (P_\lambda^\theta P_\alpha^\epsilon P_\beta^\delta \dots\dots P_1)^{\frac{1}{n}}.$$
Therefore, giving z in (24) the value λ, $R_\lambda^{\frac{1}{n}} = A_\lambda \phi_\lambda^{\frac{1}{n}}$. This is the second of equations (25). In like manner we can show that all the terms $R_1^{\frac{1}{n}}, R_\lambda^{\frac{1}{n}}, \dots\dots, R_\theta^{\frac{1}{n}}$ are to be taken as in (25). It has only to be added that $R_0^{\frac{1}{n}}$ must be taken with its rational value, because, by § 5, $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation $f(x) = 0$.

Sufficiency of the Forms.

§ 20. We here assume that R_1 has the form (20), that $R_0^{\frac{1}{n}}$ is rational, and that $R_1^{\frac{1}{n}}, R_\lambda^{\frac{1}{n}}$, etc., are taken as in (25), and we have to show that the n values of x_{s+1} in (2), obtained by giving s successively the n values $0, 1, 2, \dots\dots, n-1$, are the roots of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which they are the roots, is irreducible. *In the first place,* $R_0^{\frac{1}{n}}$ has been taken rational. *In the next place,* an equation of the type (3) subsists for every integral value of z. For, let z not be a multiple of n. In this case it may be taken to be a number in the series $1, 2, \dots\dots, n-1$. Then, by (25),
$$(R_z R_1^{-z})^{\frac{1}{n}} = (A_z A_1^{-z})(\phi_z \phi_1^{-z})^{\frac{1}{n}}. \tag{27}$$
But ϕ_1 is the expression (8). Therefore, by § 9,
$$(\phi_z \phi_1^{-z})^{\frac{1}{n}} = F(w),$$

$F(w)$ being a rational function of w. This makes (27) an equation of the type (3). Next, let z be a multiple of n, in which case it may be taken to be zero. Then

$$R_e^{\frac{1}{n}} = R_0^{\frac{1}{n}}, \text{ and } R_1^{\frac{z}{n}} = 1.$$

Therefore

$$(R_e R_1^{-e})^{\frac{1}{n}} = R_0^{\frac{1}{n}}. \tag{28}$$

Since $R_0^{\frac{1}{n}}$ is rational, (28) is an equation of the type (3). Therefore, whether z be a multiple of n or not, an equation of the type (3) subsists. *In the third place,* the equation (5) subsists along with (3) for every value of e that makes w^e a primitive n^{th} root of unity. For, let z be a multiple of n; it may be taken to be zero. Therefore

$$R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}}, \text{ and } R_e^{\frac{z}{n}} = 1.$$

Therefore

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = R_0^{\frac{1}{n}}. \tag{29}$$

But, equation (28) being regarded as (3), (29) is (5). Next, let z not be a multiple of n. It may be taken to be a number in the series $1, 2, \ldots, n-1$. Then equation (27) is (3). But, in (27), z may be any number not a multiple of n, and ez is not a multiple of n. Therefore we may substitute for z either ez or e. Thus we have

$$(R_{ez} R_1^{-ez})^{\frac{1}{n}} = (A_{ez} A_1^{-ez})(\phi_{ez}\phi_1^{-ez})^{\frac{1}{n}}$$

and

$$(R_e R_1^{-e})^{\frac{1}{n}} = (A_e A_1^{-e})(\phi_e \phi_1^{-e})^{\frac{1}{n}}.$$

Therefore

$$(R_{ez} R_e^{-z})^{\frac{1}{n}} = (A_{ez} A_e^{-z})(\phi_{ez}\phi_e^{-z})^{\frac{1}{n}}. \tag{30}$$

But, equation (27) being regarded as (3), equation (30) is (5). Therefore, whether z be a multiple of n or not, equation (5) subsists along with (3). Hence, by the Criterion in § 10, the n values of x_{s+1} in (2), obtained by giving s successively the values $0, 1, 2, \ldots, n-1$, are the roots of a pure uni-serial Abelian equation.

Particular Values of n; the Pure Abelian Cubic.

§ 21. When the equation $f(x) = 0$ is of the third degree, taking $\lambda = 2$, the series (7) is reduced to the terms 1, 2, and the equations (25) become

$$R_1^{\frac{1}{3}} = A_1 (P_1^2 P_2)^{\frac{1}{3}}, \quad R_2^{\frac{1}{3}} = A_2 (P_2^2 P_1)^{\frac{1}{3}}.$$

Also $R_0^{\frac{1}{3}} = A_0 \phi_0$. Therefore

$$x_1 = A_0\phi_0 + A_1 (P_1^2 P_2)^{\frac{1}{3}} + A_2 (P_1 P_2^2)^{\frac{1}{3}}.$$

If $A_0\phi_0 = 0$, the equation wants its second term. Then, putting

$$\psi_1 = A_1^2 A_2^{-1} P_1 \text{ and } \psi_2 = A_2^2 A_1^{-1} P_2,$$

we get

$$x_1 = (\psi_1^2\psi_2)^{\frac{1}{3}} + (\psi_2^2\psi_1)^{\frac{1}{3}}.$$

§ 22. Let the pure Abelian cubic of which x_1 is the root be

$$x^3 + px + q = 0.$$

Because ψ_1 is a rational function of the primitive third root of unity,

$$\psi_1 = b + c\sqrt{-3}$$

and

$$\psi_2 = b - c\sqrt{-3},$$

b and c being rational. Therefore $\psi_1\psi_2 = b^2 + 3c^2$. Therefore

$$x_1 = \{(b^2 + 3c^2)(b + c\sqrt{-3})\}^{\frac{1}{3}} + \{(b^2 + 3c^2)(b - c\sqrt{-3})\}^{\frac{1}{3}}.$$

But

$$x_1 = \left\{-\frac{q}{2} + \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right\}^{\frac{1}{3}} + \left\{-\frac{q}{2} - \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right\}^{\frac{1}{3}}.$$

Therefore

$$\sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)} = c\,(b^2 + 3c^2)\sqrt{-3}.$$

Therefore

$$\sqrt{(-4p^3 - 27q^2)} = 18c\,(b^2 + 3c^2).$$

Thus $\sqrt{(-4p^3 - 27q^2)}$ is rational: the well known relation between the coefficients which makes the irreducible cubic $x^3 + px + q = 0$ a pure Abelian.

The Pure Abelian Quintic.

§ 23. When $n = 5$, λ may be taken to be 2. The series (7) then becomes $1, 2, 4, 8$; or, rejecting multiples of 5, $1, 2, 4, 3$. We may then put

$$R_1^{\frac{1}{5}} = A_1\,(P_1^3 P_2^4 P_4^2 P_3)^{\frac{1}{5}},$$
$$R_2^{\frac{1}{5}} = A_2\,(P_1 P_2^3 P_4^4 P_3^2)^{\frac{1}{5}},$$
$$R_4^{\frac{1}{5}} = A_4\,(P_1^2 P_2 P_4^3 P_3^4)^{\frac{1}{5}},$$
$$R_1^{\frac{1}{5}} = A_3\,(P_1^4 P_2^2 P_4 P_3^3)^{\frac{1}{5}}.$$

If we assume R_0 to be zero,

$$x_1 = A_1\,(P_1^3 P_2^4 P_4^2 P_3)^{\frac{1}{5}} + A_2\,(P_1 P_2^3 P_4^4 P_3^2)^{\frac{1}{5}} + A_4\,(P_1^2 P_2 P_4^3 P_3^4)^{\frac{1}{5}} + A_3\,(P_1^4 P_2^2 P_4 P_3^3)^{\frac{1}{5}}. \quad (31)$$

§ 24. In a celebrated fragment (see Crelle's Journal, Vol. V, p. 336) found among the papers of Abel after his death, the root r_1 of the solvable equation of the fifth degree wanting the second term was stated, though without any accompanying demonstration, substantially as follows: Let

$$\left.\begin{aligned} a_1 &= p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})} \\ a_2 &= p - q\sqrt{z} + \sqrt{(hz - h\sqrt{z})} \\ a_4 &= p + q\sqrt{z} - \sqrt{(hz + h\sqrt{z})} \\ a_3 &= p - q\sqrt{z} - \sqrt{(hz - h\sqrt{z})} \end{aligned}\right\} \quad (32)$$

where p, q and h are rational, and

$$z = e^2 + 1, \tag{33}$$

e being rational. Then, B_1 being a rational function of α_1, B_2 the same rational function of α_2, and so on,

$$r_1 = B_1 (\alpha_1^3 \alpha_2^4 \alpha_4^2 \alpha_3)^{\frac{1}{5}} + B_2 (\alpha_1 \alpha_2^3 \alpha_4^4 \alpha_3^2)^{\frac{1}{5}} + B_4 (\alpha_1^2 \alpha_2 \alpha_4^3 \alpha_3^4)^{\frac{1}{5}} + B_3 (\alpha_1^4 \alpha_2^2 \alpha_4 \alpha_3^3)^{\frac{1}{5}}. \tag{34}$$

§ 25. The expression for r_1 in (34) is the root of a solvable irreducible quintic, not necessarily a pure Abelian. To obtain from it the necessary and sufficient form of the root of a pure Abelian quintic, we make use of the law referred to in § 5, according to which the root of the pure Abelian quintic wanting the second term is

$$R_1^{\frac{1}{5}} + R_2^{\frac{1}{5}} + R_3^{\frac{1}{5}} + R_4^{\frac{1}{5}},$$

where R_1 is a rational function of the primitive fifth root of unity w. By this law, to deduce the root x_1 of a pure Abelian quintic from the root r_1 of an irreducible solvable quintic as in (34), we have simply to pass from the more general expression α_1 to the less general expression which we have called P_1, because, in doing this, we necessarily pass from B_1 to A_1, B_1 being a rational function of α_1, and A_1 a rational function of P_1. The question, however, is: Can we pass from α_1 to P_1? In other words, can the general rational function of the primitive fifth root of unity be subsumed under α_1? That it can, may be thus shown: The value of w is

$$w = \frac{\sqrt{5} - 1}{4} + \frac{\sqrt{(-10 - 2\sqrt{5})}}{4}. \tag{35}$$

Hence, if $F(w)$ be the general rational function of w,

$$F(w) = p + k\sqrt{5} + (l + m\sqrt{5})\sqrt{(-10 - 2\sqrt{5})}, \tag{36}$$

where p, k, l and m are rational. Putting

$$z = \frac{5(l^2 + 5m^2 + 2lm)^2}{(l^2 + 5m^2 + 10lm)^2}$$

and

$$h = \frac{-2(l^2 + 5m^2 + 10lm)^2}{l^2 + 5m^2 + 2lm},$$

(36) becomes

$$F(w) = p + \frac{k(l^2 + 5m^2 + 10lm)\sqrt{z}}{l^2 + 5m^2 + 2lm} + \sqrt{(hz + h\sqrt{z})};$$

or, putting

$$q = \frac{k(l^2 + 5m^2 + 10lm)}{l^2 + 5m^2 + 2lm},$$

$$F(w) = p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}. \tag{37}$$

The value of z given above conforms to the type (33), for it can be changed into

$$z = \left\{ \frac{2(l^2 - 5m^2)}{l^2 + 5m^2 + 10lm} \right\}^2 + 1 .$$

Hence the general rational function of the primitive fifth root of unity falls under the expression for α_1 in (32).

§ 26. The writer may perhaps be permitted to refer to a paper of his, entitled "Solution of Solvable Irreducible Quintic Equations," which appeared in this Journal, Vol. VII, No. 2. Assuming that the quintic to be solved has, by Jerrard's application of the method of Tschirnhaus, been brought to the trinomial form

$$x^5 + px + q = 0 , \tag{38}$$

he proved, in the article referred to, that it admits of algebraical solution only if

$$p = \frac{5A^4(3 - B)}{16 + B^2}$$

and

$$q = \frac{A^5(22 + B)}{16 + B^2} .$$

When the coefficients are thus related, take λ a root of the equation

$$x^4 - Bx^3 - 6x^2 + Bx + 1 = 0 .$$

Put

$$a = \frac{-(\lambda^2 + 1)}{A\lambda(\lambda - 1)}$$

and

$$\theta = \frac{-A^5\lambda(\lambda - 1)^2}{(16 + B^2)(\lambda + 1)(\lambda^2 + 1)} ;$$

then the solution of the equation (38) is

$$r_1 = \theta^{\frac{1}{5}} + a\theta^{\frac{2}{5}} + \lambda a^2\theta^{\frac{3}{5}} - \lambda a^3\theta^{\frac{4}{5}} .$$

This form of the root may at first sight seem to have no affinity with the Abelian form in (34); but, in a communication which was laid before the Royal Society of Canada at its meeting in May, 1886, and which is to appear in the forthcoming volume of the Transactions of the Society, the writer has shown the essential identity of the two forms.

The Pure Uni-Serial Abelian Quartic.

Necessary and Sufficient Forms of the Roots.

§ 27. Taking $z = e^2 + 1$ as in (33), the necessary and sufficient forms of the roots of the pure uni-serial Abelian quartic are the expressions α_1, α_2, α_4, α_3 in

(32); the rational expressions p, q, h, e being subject to the sole restriction that they must leave the equation of the fourth degree, which has $\alpha_1, \alpha_2, \alpha_4$ and α_3 for its roots, irreducible. There is thus an intimate relation between the pure uni-serial Abelian of the fourth degree and the solvable irreducible equation of the fifth degree. This is only a case of a more general law. If $2n+1$ be any prime number, and if the forms of the roots of the pure uni-serial Abelian of degree $2n$ have been found, the necessary and sufficient forms of the roots of the solvable irreducible equation of degree $2n+1$ can be found.

Necessity of the Forms (32).

§ 28. Here an equation of the fourth degree $f(x)=0$ is assumed to be a pure uni-serial Abelian; and we have to show that its roots are of the forms $\alpha_1, \alpha_2, \alpha_4, \alpha_3$ in (32). The roots of the equation $f(x)=0$, in the familiar Abelian notation, are

$$x_1,\ \theta x_1,\ \theta^2 x_1,\ \theta^3 x_1. \tag{39}$$

Because x_1 is the root of an irreducible quartic, its form is

$$x_1 = P + \sqrt{Q},$$

where P is clear of the radical $\sqrt{Q}$. Another root of the quartic is $P-\sqrt{Q}$. This is obtained from x_1 by changing the sign of $\sqrt{Q}$; and, by changing the sign of $\sqrt{Q}$ in $P-\sqrt{Q}$, we return to $P+\sqrt{Q}$ or x_1. Hence $P-\sqrt{Q}$ must be the third term in (39). Therefore

$$\theta^2 x_1 = P - \sqrt{Q}.$$

In passing from x_1 to θx_1, let P and Q become P' and Q' respectively; then

$$\theta x_1 = P' + \sqrt{Q'};$$

therefore

$$\theta^3 x_1 = P' - \sqrt{Q'}.$$

In running through the series (39), the root of the equation $f(x)=0$ undergoes all its possible changes. But, from the expressions that have been obtained for $x_1, \theta x_1, \theta^2 x_1$ and $\theta^3 x_1$, P can take only the two values P, P', and Q can take only the two values Q, Q'. Therefore each of the expressions P and Q is the root of a quadratic equation. Hence the only radicals occurring in x_1 are square roots. But, when square roots are the only radicals in the root of an equation of the fourth degree, its root must be either

$$\left.\begin{array}{l} p+\sqrt{s}+\sqrt{t}, \\ \text{or} \quad p+k\sqrt{s}+\sqrt{(l+m\sqrt{s})} \end{array}\right\} \tag{40}$$

p, s, t, k, l and m being rational. Suppose, if possible, that x_1 is of the first of the forms (40); then either

$$\theta x_1 = p + \sqrt{s} - \sqrt{t} \therefore \theta^2 x_1 = p + \sqrt{s} + \sqrt{t} = x_1,$$

or

$$\theta x_1 = p - \sqrt{s} + \sqrt{t} \therefore \theta^2 x_1 = p + \sqrt{s} + \sqrt{t} = x_1,$$

or

$$\theta x_1 = p - \sqrt{s} - \sqrt{t} \therefore \theta^2 x_1 = p + \sqrt{s} + \sqrt{t} = x_1.$$

But the equation $f(x) = 0$, being a pure Abelian, is irreducible, and therefore cannot have equal roots. Therefore x_1 is not of the first of the forms (40). It is therefore of the second. Consequently we may put

$$\left.\begin{aligned} x_1 &= p + k\sqrt{s} + \sqrt{(l + m\sqrt{s})} \\ \theta x_1 &= p - k\sqrt{s} + \sqrt{(l - m\sqrt{s})} \\ \theta^2 x_1 &= p + k\sqrt{s} - \sqrt{(l + m\sqrt{s})} \\ \theta^3 x_1 &= p - k\sqrt{s} - \sqrt{(l - m\sqrt{s})} \end{aligned}\right\} \qquad (41)$$

It is plain that $\theta^2 x_1$ must have the place assigned to it in (41), because the change that causes x_1 to become $\theta^2 x_1$ must transform $\theta^2 x_1$ into x_1. We can now determine the expression $\sqrt{(l + m\sqrt{s})}$ more definitely. To pass from x_1 to θx_1 we change the sign of $\sqrt{s}$ and take the resulting radical $\sqrt{(l - m\sqrt{s})}$ with the positive sign. In order that these changes may cause θx_1 to become $\theta^2 x_1$, the changes must admit of being made on θx_1. In other words, the radical $\sqrt{(l - m\sqrt{s})}$, which does not occur in that form in x_1, must be expressible in terms of the radicals in x_1. Therefore we must have

$$\sqrt{(l - m\sqrt{s})} = (c + d\sqrt{s}) + (g - r\sqrt{s})\sqrt{(l + m\sqrt{s})},$$

c, d, g and r being rational. Therefore

$$l - m\sqrt{s} = (c + d\sqrt{s})^2 + (g - r\sqrt{s})^2(l + m\sqrt{s}) + 2(c + d\sqrt{s})(g - r\sqrt{s})\sqrt{(l + m\sqrt{s})}.$$

Hence $(c + d\sqrt{s})(g - r\sqrt{s})$ must be zero; for, if it were not, $\sqrt{(l + m\sqrt{s})}$ would be a rational function of $\sqrt{s}$, which would make x_1 in (41) the root of a quadratic. And $g - r\sqrt{s}$ cannot be zero, for this would make

$$\sqrt{(l - m\sqrt{s})} = c + d\sqrt{s},$$

and therefore, by (41), θx_1 would be the root of a quadratic. Hence $c + d\sqrt{s}$ is zero, and therefore

$$\sqrt{(l - m\sqrt{s})} = (g - r\sqrt{s})\sqrt{(l + m\sqrt{s})}. \qquad (42)$$

By comparing the first three of equations (41) with one another, it appears that the change which transforms $\sqrt{(l + m\sqrt{s})}$ into $\sqrt{(l - m\sqrt{s})}$ causes $\sqrt{(l - m\sqrt{s})}$ to become $-\sqrt{(l + m\sqrt{s})}$. Consequently, from (42),

$$-\sqrt{(l + m\sqrt{s})} = (g + r\sqrt{s})\sqrt{(l - m\sqrt{s})}. \qquad (43)$$

From (42) and (43),

$$g^2 - r^2 s = -1 \therefore \sqrt{s} = \frac{\sqrt{(g^2+1)}}{r}. \tag{44}$$

By squaring both sides of (43) and equating the parts involving the radical $\sqrt{s}$,

$$2grl = m(1 + g^2 + r^2 s).$$

Therefore, by (44), $$2grl = 2m\,(1 + g^2).$$

$$\therefore\quad l = \frac{m}{gr}\,(1 + g^2).$$

Substitute in the first of equations (41) this value of l, substituting at the same time for $\sqrt{s}$ its value in (44). Then, writing z for $1 + \left(\frac{1}{g}\right)^2$ and h for $\frac{mg}{r}$, and q for $\frac{kg}{r}$,

$$x_1 = p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}.$$

Thus the necessity of the forms in (32) is established.

Sufficiency of the Forms.

§ 29. We now take $\alpha_1, \alpha_2, \alpha_4, \alpha_3$, as in (32), subject to the restriction that the quartic equation of which they are the roots must be irreducible, and we have to show that this equation is a pure uni-serial Abelian. The radical $\sqrt{(hz - h\sqrt{z})}$, which occurs in α_2, is not found in that form in α_1. But, keeping in view that $z = e^2 + 1$,

$$\sqrt{(hz - h\sqrt{z})} = \frac{\sqrt{z} - 1}{e}\sqrt{(hz + h\sqrt{z})}. \tag{45}$$

It is obvious that the expression

$$p - q\sqrt{z} + \frac{\sqrt{z}-1}{e}\sqrt{(hz + h\sqrt{z})}$$

is a rational function of the expression

$$p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}.$$

Therefore α_2 is a rational function of α_1; and the equation $f(x) = 0$ is a pure Abelian. That it is uni-serial may be thus shown. To pass from α_1 to α_2, we change the sign of $\sqrt{z}$, and take the resulting radical $\sqrt{(hz - h\sqrt{z})}$ with the positive sign. Let these same changes be made on α_2. The result, by (45), is

$$p + q\sqrt{z} - \frac{\sqrt{z}+1}{e}\sqrt{(hz - h\sqrt{z})}.$$

And this again, by (45), is equivalent to

$$p + q\sqrt{z} - \frac{\sqrt{z}+1}{e}\,\frac{\sqrt{z}-1}{e}\sqrt{(hz + h\sqrt{z})},$$

which, because $z = e^2 + 1$, is

$$p + q\sqrt{z} - \sqrt{(hz + h\sqrt{z})}, \text{ or } \alpha_4.$$

Hence, in passing from α_1 to α_2, we pass from α_2 to α_4; and in like manner it may be shown that the same changes of the radicals carry us from α_4 to α_3 and from α_3 back to α_1; consequently the pure Abelian equation $f(x) = 0$ is uni serial.

The Fundamental Element of the Root.

§30. The problem of the necessary and sufficient forms of the roots of the pure uni-serial Abelian equation of the fourth degree has been solved. We propose to find the solution by another method; and, with a view to a comparison of the result obtained above with that at which we shall arrive by the second method, we may now find expressions for R_1, the fundamental element of the root, and for the derived expressions R_0, R_2, R_3.

§31. By §5 the four roots of the pure uni-serial Abelian quartic equation $f(x) = 0$ are

$$\begin{aligned} x_1 &= R_0^{\frac14} + R_1^{\frac14} + R_2^{\frac14} + R_3^{\frac14}, \\ \theta x_1 = x_2 &= R_0^{\frac14} + wR_1^{\frac14} + w^2R_2^{\frac14} + w^3R_3^{\frac14}, \\ \theta^2 x_1 = x_4 &= R_0^{\frac14} + w^2R_1^{\frac14} + R_2^{\frac14} + w^2R_3^{\frac14}, \\ \theta^3 x_1 = x_3 &= R_0^{\frac14} + w^3R_1^{\frac14} + w^2R_2^{\frac14} + wR_3^{\frac14}, \end{aligned}$$

w being a primitive fourth root of unity. Therefore, because $w^2 = -1$, and $w^3 = -w$,

$$\left.\begin{aligned} 4R_0^{\frac14} &= x_1 + x_2 + x_4 + x_3 \\ 4R_1^{\frac14} &= x_1 + w^3x_2 + w^2x_4 + wx_3 = (x_1 - x_4) - w(x_2 - x_3) \\ 4R_2^{\frac14} &= x_1 + w^2x_2 + x_4 + w^2x_3 = (x_1 + x_4) - (x_2 + x_3) \\ 4R_3^{\frac14} &= x_1 + wx_2 + w^2x_4 + w^3x_3 = (x_1 - x_4) + w(x_2 - x_3) \end{aligned}\right\} \quad (46)$$

But, by what was proved above,

$$\begin{aligned} x_1 &= p + q\sqrt{z} + \sqrt{(hz + h\sqrt{z})}, \\ x_2 &= p - q\sqrt{z} + \sqrt{(hz - h\sqrt{z})}, \\ x_4 &= p + q\sqrt{z} - \sqrt{(hz + h\sqrt{z})}, \\ x_3 &= p - q\sqrt{z} - \sqrt{(hz - h\sqrt{z})}. \end{aligned}$$

Therefore, by (46),

$$\begin{aligned} R_0^{\frac14} &= p, \\ 2R_1^{\frac14} &= \sqrt{(hz + h\sqrt{z})} - w\sqrt{(hz - h\sqrt{z})}, \\ R_2^{\frac14} &= q\sqrt{z}, \\ 2R_3^{\frac14} &= \sqrt{(hz + h\sqrt{z})} + w\sqrt{(hz - h\sqrt{z})}. \end{aligned}$$

Therefore, keeping in view that $z = e^2 + 1$, and making use of the relation $\sqrt{(hz + h\sqrt{z})}\sqrt{(hz - h\sqrt{z})} = he\sqrt{z}$,

$$\left.\begin{aligned} R_0 &= p^4 \\ 4R_1 &= h^2(e^2+1)(we-1)^2 \\ R_2 &= q^4z^2 \\ 4R_3 &= h^2(e^2+1)(we+1)^2 \end{aligned}\right\} \tag{47}$$

§ 32. It may not be out of place to observe that, in (47), R_1 is not presented in the form in which it is a fundamental element of the root of the pure uni-serial Abelian quartic equation $f(x) = 0$; that is to say, it is not in the form in which R_0, R_2 and R_3 can be derived from it by changing w into w^0, w^2 and w^3 respectively. In fact, by changing w in R_1, as given in (47), into w^2, we should obtain $\frac{1}{4}h^2(e^2+1)(e+1)^2$; whereas, by (47), R_2 is q^4z^2 or $q^4(e^2+1)^2$. The form of R_1, in which it is the fundamental element of a root of a pure uni-serial Abelian quartic, will be determined afterwards.

The Problem of the Necessary and Sufficient Forms of the Roots of the Pure Uni-Serial Abelian Quartic Solved from Another Point of View.

The Fundamental Element of the Root.

§ 33. The necessary and sufficient forms of the roots of the pure uni-serial Abelian equation of the fourth degree may be found in another manner; namely, by making use of the principles laid down in § 5, so as to determine the fundamental element R_1 of the root. Let w be a primitive fourth root of unity. Take any rational quantities, b, c, d, m. Find the rational quantities, p, q, r, s, by means of the three equations, equivalent to four linear equations,

$$\left.\begin{aligned} p+q+r+s \quad &= d^4 \\ p-q+r-s \quad &= \frac{m^4}{(b^2+c^2)^2} \\ (p-r)+w(q-s) &= \frac{m^2(b+cw)^2}{b^2+c^2} \end{aligned}\right\}. \tag{48}$$

Then it will be found that

$$R_1 = p + qw + rw^2 + sw^3. \tag{49}$$

The Root Constructed from its Fundamental Element.

§34. Having found R_1 as in (49), derive from it R_0, R_2, R_3 by changing w into w^0, w^2, w^3 respectively. But, since each of the expressions $R_1^{\frac{1}{4}}$, $R_2^{\frac{1}{4}}$, etc., has four values for given values of R_1, R_2, etc., we must settle what values of these expressions are to be taken together in order that

$$R_0^{\frac{1}{4}} + R_1^{\frac{1}{4}} + R_2^{\frac{1}{4}} + R_3^{\frac{1}{4}} \tag{50}$$

may be the root of a pure uni-serial Abelian quartic. From the two equations

$$R_1 = (p - r) + w(q - s) = \frac{m^2(b + cw)^2}{b^2 + c^2},$$

$$R_3 = (p - r) - w(q - s) = \frac{m^2(b - cw)^2}{b^2 + c^2},$$

it follows that $R_1 R_3 = m^4$. Having taken $R_1^{\frac{1}{4}}$ a definite fourth root of unity, take $R_3^{\frac{1}{4}}$ such that

$$R_1^{\frac{1}{4}} R_3^{\frac{1}{4}} = m. \tag{51}$$

Then, because $R_0 = p + q + r + s = d^4$, take $R_0^{\frac{1}{4}}$ such that

$$R_0^{\frac{1}{4}} = d. \tag{52}$$

Finally, because $R_2 = p - q + r - s = \frac{m^4}{(b^2 + c^2)^2}$, let $R_2^{\frac{1}{4}}$ be such that $R_2^{\frac{1}{4}}$ is positive. The expressions $R_0^{\frac{1}{4}}$, $R_1^{\frac{1}{4}}$, etc., being thus determined, the expression (50) shall be the root of a pure uni-serial Abelian equation of the fourth degree, provided always that the equation of the fourth degree, of which it can be shown to be a root, is irreducible.

Necessity of the Above Forms.

§35. Here we assume x_1 to be the root of a pure uni-serial Abelian quartic equation $f(x) = 0$. By §5,

$$x_1 = R_0^{\frac{1}{4}} + R_1^{\frac{1}{4}} + R_2^{\frac{1}{4}} + R_3^{\frac{1}{4}}; \tag{53}$$

and what we have to make out is that R_1 has the form given in (49), and that $R_1^{\frac{1}{4}}$ and $R_3^{\frac{1}{4}}$ are related in such a manner that the equation (51) subsists, while $R_2^{\frac{1}{4}}$ is essentially positive. When we say that R_1 has the form given in (49), it is understood that p, q, r and s are determined by the equations (48).

§36. Because $F(w)$ in (3) is a rational function of w, we may put

$$F(w) = (b + cw)^{-1},$$

b and c being rational. Therefore, from (3), taking $z=2$,

$$R_2^{\frac{1}{4}} = (b+cw)^{-1}R_1^{\frac{1}{2}}. \tag{54}$$

Therefore, by (5), taking $e=3$,

$$R_2^{\frac{1}{4}} = (b-cw)^{-1}R_3^{\frac{1}{2}}.$$

Therefore

$$\left.\begin{aligned} R_2^{\frac{1}{2}} &= (b^2+c^2)^{-1}(R_1R_3)^{\frac{1}{2}} \\ \therefore\ R_2 &= (b^2+c^2)^{-2}(R_1R_3) \end{aligned}\right\} \tag{55}$$

But R_1 is a rational function of w. We may put $R_1 = t+\tau w$ and $R_3 = t-\tau w$, t and τ being rational. Therefore R_1R_3 is equal to the positive quantity $t^2+\tau^2$. Therefore, from the second of equations (55), R_2 is positive.

§37. Because $b+cw$ and R_1 are rational functions of w, we may put

$$(b+cw)^{-2}R_1 = d+\delta w,$$

d and δ being rational. Therefore, from (54),

$$R_2 = \{(b+cw)^{-2}R_1\}^2 = d^2-\delta^2+2d\delta w.$$

Since R_2 is rational, $d\delta = 0$. And δ must be zero; for, if it were not, d would be zero, and we should have $R_2 = -\delta^2$, which, because R_2 has been shown to be positive, is impossible. Therefore

$$\left.\begin{aligned} (b+cw)^{-2}R_1 &= d \\ \text{Therefore also}\qquad (b-cw)^{-2}R_3 &= d \end{aligned}\right\} \tag{56}$$

Therefore

$$R_3R_1^{-3} = \{d\,(b+cw)^2\}^{-4}\{d\,(b^2+c^2)\}^2.$$

From (3), $R_3R_1^{-3}$ is the fourth power of a rational function of w. Therefore $\{d\,(b^2+c^2)\}^2$ is the fourth power of a rational function of w. Therefore

$$\pm\, d\,(b^2+c^2) = (g+kw)^2 = g^2-k^2+2gkw,$$

g and k being rational, the double sign on the extreme left of the equation indicating that it is not yet determined which of the two signs is to be taken. Hence $gk=0$. Therefore $\pm\, d\,(b^2+c^2)$ is equal either to g^2 or to $-k^2$. That is, $d\,(b^2+c^2)$ is the square of a rational quantity, with the positive or negative sign. Hence we may put

$$d\,(b^2+c^2) = m^2w^{2s},$$

m being rational and w^{2s} having one of the two values $1, -1$. Substituting for d in (56) its value now obtained,

$$R_1 = \frac{m^2w^{2s}(b+cw)^2}{b^2+c^2},$$

and

$$R_3 = \frac{m^2w^{2s}(b-cw)^2}{b^2+c^2}.$$

But w^{2s} is either 1 or -1. In the former case,

$$R_1 = \frac{m^2(b+cw)^2}{(b^2+c^2)}. \tag{57}$$

In the latter case, $w^{2s} = w^2$. Then

$$R_1 = \frac{m^2(bw-c)^2}{b^2+c^2},$$

an expression essentially of the same character as (57). Therefore (57) is the universal form of R_1. From (57),

$$R_3 = \frac{m^2(b-cw)^2}{b^2+c^2}.$$

Therefore $R_1R_3 = m^4$. Hence, from (55),

$$R_2 = \frac{m^4}{(b^2+c^2)^2}. \tag{58}$$

Let R_1, when so expressed that it is the fundamental element of the root of a pure uni-serial Abelian quartic, be

$$R_1 = p' + q'w + r'w^2 + s'w^3 = (p'-r') + w(q'-s'),$$

p', q', r' and s' being rational. Then

$$R_2 = p' + q'w^2 + r' + s'w^2 = (p'+r') - (q'+s').$$

Therefore, by (57) and (58),

$$\left.\begin{aligned} (p'+r') - (q'+s') &= \frac{m^4}{(b^2+c^2)^2} \\ \text{and} \quad (p'-r') + w(q'-s') &= \frac{m^2(b+cw)^2}{b^2+c^2} \end{aligned}\right\} \tag{59}$$

And, by §5, $R_0^{\frac{1}{n}}$ is rational. Therefore, d being some rational quantity,

$$p' + q' + r' + s' = d^4. \tag{60}$$

The equations (59) and (60) for the determination of p', q', r', s' are the same as the equations (48) for the determination of p, q, r, s. Therefore

$$p' = p,\ q' = q,\ r' = r,\ s' = s.$$

Hence

$$R_1 = p + qw + rw^2 + sw^3,$$

which is the form of the fundamental element in (49). And, by §34, in constructing the root x_1 from its fundamental element, having assigned a definite character to $R_1^{\frac{1}{4}}$, we then, knowing that R_1R_3 is equal to m^4, selected the value of $R_3^{\frac{1}{4}}$ so as to make $R_1^{\frac{1}{4}}R_3^{\frac{1}{4}}$ equal to m. Hence the necessity of the form of R_1 in (49) and of the relation between the roots $R_1^{\frac{1}{4}}$ and $R_3^{\frac{1}{4}}$ indicated in (51) is made

good. At the same time, because $R_1^{\frac{1}{4}}R_3^{\frac{1}{4}} = m$, $R_1^{\frac{1}{4}}R_3^{\frac{3}{4}} = m^2$; therefore, by the first of equations (55), $R_3^{\frac{1}{4}}$ is positive.

Sufficiency of the Forms.

§ 38. To prove that the above forms are sufficient, we have to show that the conditions specified in § 10 are satisfied, it being assumed that the equation of the fourth degree, of which the root is given in (53), is irreducible. *The first condition* is that $R_0^{\frac{1}{n}}$ must be rational. This is satisfied by the first of equations (48). *The next condition* is that an equation of the type (3) subsists for every integral value of z. It will be enough to consider two values of z, namely, 2 and 3. Because

$$R_1 = p + qw + rw^2 + sw^3 = (p - r) + w(q - s)$$

and

$$R_2 = p + qw^2 + r \quad + sw^2 = (p + r) - \quad (q + s),$$

we have, from the last two of equations (48),

$$R_2 R_1^{-2} = \frac{m^4}{(b^2 + c^2)^2} \times \frac{m^{-4}}{(b^2 + c^2)^{-2}} (b + cw)^{-4} = (b + cw)^{-4}.$$

Hence an equation of the type (3) subsists when $z = 2$. Again,

$$R_3 = p + qw^3 + rw^2 + sw = (p - r) - w(q - s).$$

But

$$(p - r) + w(q - s) = \frac{m^2 (b + cw)^2}{b^2 + c^2};$$

therefore

$$(p - r) - w(q - s) = \frac{m^2 (b - cw)^2}{b^2 + c^2}.$$

Therefore

$$R_3 R_1^{-3} = m^{-4} (b + cw)^4 (b - cw)^{-4}.$$

Hence an equation of the type (3) subsists when $z = 3$. Consequently an equation of the type (3) subsists for every integral value of z. *The third condition* is that equation (5) must subsist along with (3) for every value of e prime to 4. As we may leave out of view values of e greater than 4, we have only to consider the case in which $e = 3$. Also it will be enough to consider the cases in which z is equal to one of the numbers 0, 2, 3. Let $z = 0$. Then equation (3) is

$$R_0^{\frac{1}{4}} = \{F(w)\} R_1^{\frac{0}{4}} = F(w).$$

But $R_0^{\frac{1}{4}}$ is rational. Hence, changing w into w^3,

$$R_0^{\frac{1}{4}} = F(w^3).$$

Also $R_{ez}^{\frac{1}{4}} = R_0^{\frac{1}{4}}$. Therefore

$$R_{ez}^{\frac{1}{4}} = F(w^3) = \{F(w^3)\} R_3^{\frac{0}{4}}.$$

This is equation (5); so that, when $z=0$, equation (5) subsists along with (3). Next, let $z=2$. Then equation (3) is

$$R_2^{\frac{1}{4}} = \{F(w)\} R_1^{\frac{2}{4}}. \tag{61}$$

$$\therefore\ R_2 = \{F(w)\}^4 R_1^2.$$

Therefore, changing w into w^3,

$$R_2 = \{F(w^3)\}^4 R_3^2.$$

Therefore
$$R_2^{\frac{1}{4}} = w'\{F(w^3)\} R_3^{\frac{2}{4}}, \tag{62}$$

w' being an n^{th} root of unity. From (61) and (62),

$$R_2^{\frac{1}{2}} = w'\{F(w)\}\{F(w^3)\}(R_1R_3)^{\frac{1}{2}}.$$

Let $F(w) = g + hw$, g and h being rational. Therefore $F(w^3) = g - hw$. Therefore $\{F(w)\}\{F(w^3)\}$ is equal to the positive quantity $g^2 + h^2$. Also, from the manner in which the root x_1 was constructed in § 34 from its fundamental element, $R_1^{\frac{1}{4}} R_3^{\frac{1}{4}} = m$. Therefore $(R_1R_3)^{\frac{1}{2}}$ is positive. Also, in constructing the root, $R_2^{\frac{1}{2}}$ was taken positive. Therefore w' is positive; that is, $w' = 1$. Therefore, from (62),

$$R_2^{\frac{1}{4}} = \{F(w^3)\} R_3^{\frac{2}{4}}. \tag{63}$$

But, equation (61) being (3), (63) is (5); so that, when $z=2$, equation (5) subsists along with (3) Finally, let $z=3$. Then equation (3) is

$$R_3^{\frac{1}{4}} = q_1 R_1^{\frac{3}{4}}, \tag{64}$$

q_1 being a rational function of w. Therefore

$$R_3 = q_1^3 R_1^3.$$

Therefore, changing w into w^3, and denoting by q_3 what q_1 becomes when w is changed into w^3,

$$R_1 = q_3^4 R_3^3.$$

Therefore
$$R_1^{\frac{1}{4}} = w' q_3 R_3^{\frac{3}{4}}, \tag{65}$$

w' being one of the fourth roots of unity. From (64) and (65),

$$(R_1R_3)^{\frac{1}{2}}(q_1q_3)\,w' = 1.$$

But in the same way in which the product of $F(w)$ and $F(w^3)$ was shown to be positive, q_1q_3 can be shown to be positive. Also $(R_1R_3)^{\frac{1}{4}} = m$. Therefore $(R_1R_3)^{\frac{1}{2}} = m^2$. Hence w' must be positive. Therefore $w' = 1$, and (65) becomes

$$R_1^{\frac{1}{4}} = q_3 R_3^{\frac{3}{4}}. \tag{66}$$

Equation (64) being (3), equation (66) is (5). Hence, whether z be zero, or 2 or 3, equation (5) subsists along with (3). Thus all the conditions specified in § 12 are satisfied, and hence, by the Criterion in § 10, x_1 is the root of a pure uni-serial Abelian quartic.

Identity of the Results Obtained by the Two Methods.

§39. It may be well to show that the results obtained by the two methods that have been employed for finding the necessary and sufficient forms of the roots of the pure uni-serial Abelian equation of the fourth degree are identical. In (47) we have expressions for R_1, R_2 and R_3 as determined by the first method. What we need to make out is that these are substantially the same as the expressions for R_1, R_2 and R_3 obtained by the second method. By (48),

$$R_1 = \frac{m^2(b+cw)^2}{b^2+c^2}.$$

Write $\frac{h}{2}$ for $\frac{mb^2}{b^2+c^2}$ and $-e$ for $\frac{c}{b}$. Then

$$\frac{m^2}{b^2+c^2} = \frac{h^2(b^2+c^2)}{4b^4}.$$

Also $\frac{b^2+c}{b^2} = 1-e^2$ and $\frac{2bc}{b^2} = -2e$. Therefore

$$b^2-c^2+2bcw = b^2(1-e^2-2ew);$$

or

$$(b+cw)^2 = b^2(1-ew)^2.$$

Therefore $$\frac{m^2(b+cw)^2}{b^2+c^2} = \frac{h^2(b^2+c^2)}{4b^2}(1-ew)^2 = \frac{h^2}{4}(1+e^2)(1-ew)^2.$$

The expression on the extreme left of this result is the value of R_1 obtained by the second method, while that on the extreme right is the value of R_1 obtained by the first method. The value of R_3 by either method is what R_1 becomes by changing w into w^3 or $-w$; so that, when the identity of the expressions obtained for R_1 by the two methods has been established, the identity of the expressions for R_3 follows. Finally, by the second of equations (48),

$$R_2 = \frac{m^4}{(b^2+c^2)^2}.$$

The above values of h and e make this

$$R_2 = \frac{h^4}{16b^4}(1+e^2)^2.$$

Put z for $1+e^2$, and q for $\frac{h}{2b}$. Then

$$R_2 = q^4z^2,$$

which is the expression for R_2 in (47).

The Pure Uni-Serial Abelian of a Degree which is the Continued Product of a Number of Distinct Prime Numbers.

Fundamental Element of the Root.

§40. Let n be the continued product of the distinct prime numbers

$$s,\ t,\ \ldots\ldots,\ d,\ b. \tag{67}$$

Take w a primitive n^{th} root of unity. Then, if

$$\sigma,\ \tau,\ \ldots\ldots,\ \delta,\ \beta \tag{68}$$

be such that $n = s\sigma = t\tau = \ldots\ldots = b\beta$, w^σ is a primitive s^{th} root of unity, w^τ a primitive t^{th} root of unity, and so on. Let

$$\left.\begin{array}{l} w^\sigma,\ w^{\sigma\lambda},\ w^{\sigma\lambda^2},\ \ldots\ldots,\ w^{\sigma\lambda^{s-2}} \\ w^\tau,\ w^{\tau h},\ w^{\tau h^2},\ \ldots\ldots,\ w^{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w^\delta,\ w^{\delta l},\ w^{\delta l^2},\ \ldots\ldots,\ w^{\delta l^{d-2}} \\ w^\beta,\ w^{\beta k},\ w^{\beta k^2},\ \ldots\ldots,\ w^{\beta k^{b-2}} \end{array}\right\} \tag{69}$$

be cycles containing respectively all the primitive s^{th} roots of unity, all the primitive t^{th} roots of unity, and so on. Should the numbers forming the series (67) be all odd, each of the cycles (69) consists of more terms than one. Should the prime number 2 be a term in (67), say b, the last of the cycles (69) would be reduced to the single term w^β, which it will be convenient to regard as a cycle though it consists of only one term. In this case $k = 1$. It may be assumed that λ is less than s, h less than t, and so on as regards all the numbers s, t, etc., in (67) which are odd primes. The numbers λ, h, etc., are prime roots of s, t, etc., respectively. Take P_1 a rational function of w, and, z being any integer, let P_z be what P_1 becomes when z is changed into w^z. Put

$$\left.\begin{array}{l} \phi_\sigma = P_\sigma^{\lambda^{s-2}} P_{\sigma\lambda}^{\lambda^{s-3}} P_{\sigma\lambda^2}^{\lambda^{s-4}} \ldots\ldots P_{\sigma\lambda^{s-2}} \\ \psi_\tau = P_\tau^{h^{t-2}} P_{\tau h}^{h^{t-3}} P_{\tau h^2}^{h^{t-4}} \ldots\ldots P_{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_\delta = P_\delta^{l^{d-2}} P_{\delta l}^{l^{d-3}} P_{\delta l^2}^{l^{d-4}} \ldots\ldots P_{\delta l^{d-2}} \\ F_\beta = P_\beta^{k^{b-2}} P_{\beta k}^{k^{b-3}} P_{\beta k^2}^{k^{b-4}} \ldots\ldots P_{\beta k^{b-2}} \end{array}\right\} \tag{70}$$

In the case when one of the numbers in (67), say b, is 2, the last of equations (70) is reduced to

$$F_\beta = P_\beta. \tag{71}$$

Then, if R_1 be the fundamental element of the root of a pure uni-serial Abelian equation $f(x) = 0$ of the n^{th} degree, it will be found that

$$R_1 = A_1^n (\phi_\sigma^\sigma \psi_\tau^\tau \ldots\ldots X_\delta^\delta F_\beta^\beta), \tag{72}$$

A_1 being a rational function of w.

The Root Constructed from its Fundamental Element.

§41. From R_1, as expressed in (72), derive R_0, R_2, etc., by changing w into w^0, w^2, etc. By §5, the root of the equation $f(x) = 0$ is

$$R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \dots + R_{n-1}^{\frac{1}{n}}. \tag{73}$$

To construct the root, we have to determine the particular n^{th} roots of R_0, R_1, etc., that are to be taken together in (73). When w is changed into w^z, let A_1, ϕ_1, ψ_1, etc., become A_z, ϕ_z, ψ_z, etc., respectively. Then

$$\left.\begin{aligned} R_z &= A_z^n(\phi_{z\sigma}^{\sigma}\psi_{z\tau}^{\tau} \dots X_{z\delta}^{\delta}F_{z\beta}^{\beta}) \\ \text{therefore} \qquad R_z^{\frac{1}{n}} &= w' A_z(\phi_{z\sigma}^{\sigma}\psi_{z\tau}^{\tau} \dots X_{z\delta}^{\delta}F_{z\beta}^{\beta})^{\frac{1}{n}} \end{aligned}\right\} \tag{74}$$

w' being an n^{th} root of unity. Let the integers not greater than n that measure n, unity not included, be

$$n,\ y,\ \text{etc.} \tag{75}$$

For instance, if $n = 3 \times 5 \times 7 = 105$, the series (75) is

$$105,\ 35,\ 21,\ 15,\ 7,\ 5,\ 3.$$

The n^{th} roots of unity distinct from unity are the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so on. For instance, the series of the 105th roots of unity distinct from unity, containing 104 terms, is made up of the 48 primitive 105th roots of unity, the 24 primitive 35th roots of unity, the 12 primitive 21st roots of unity, the 8 primitive 15th roots of unity, the 6 primitive 7th roots of unity, the 4 primitive 5th roots of unity, and the 2 primitive 3d roots of unity. The general primitive n^{th} root of unity being w^e, give w' in the second of equations (74) the value unity for every value of z included under e. Then

$$R_e^{\frac{1}{n}} = A_e(\phi_{e\sigma}^{\sigma}\psi_{e\tau}^{\tau} \dots X_{e\delta}^{\delta}F_{e\beta}^{\beta})^{\frac{1}{n}}. \tag{76}$$

Taking any other term than n, say y, in the series (75), since y is a factor of n, let $yv = n$. Then w^v is a primitive y^{th} root of unity. Hence, since w^e is the general primitive n^{th} root of unity, all the primitive y^{th} roots of unity are included in w^{ev}. If w', in the second of equations (74), be w^a when $z = v$, let it have the value w^{ea} when $z = ev$. Then

$$R_{ev}^{\frac{1}{n}} = w^{ea}A_{ev}(\phi_{ev\sigma}^{\sigma}\psi_{ev\tau}^{\tau} \dots X_{ev\delta}^{\delta}F_{ev\beta}^{\beta})^{\frac{1}{n}}. \tag{77}$$

Form equations similar to (77) for the remaining terms in (75). In this way, because the series of the n^{th} roots of unity distinct from unity is made up of the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so forth, all the terms $1, 2, \dots, n-1$ are found in the groups of numbers represented

by the subscripts e, ev, etc., with multiples of n rejected. Consequently, in determining $R_e^{\frac{1}{n}}$, $R_{ev}^{\frac{1}{n}}$, etc., as in (76), (77), etc., we have determined all the terms

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ \ldots\ldots,\ R_{n-1}^{\frac{1}{n}}. \tag{78}$$

Substitute, then, in (73) the rational value which $R_0^{\frac{1}{n}}$ can be shown, as in § 8, to possess, and the values of the terms in (78) as these are determined in (76), (77), etc., and the root is constructed; in other words, the expression (73) shall be the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible.

Necessity of the Above Forms.

§ 42. Here we assume that the root of a pure uni-serial Abelian equation $f(x)=0$ of the n^{th} degree is expressible as in (73), and we have to prove that its fundamental element R_1 has the form (72), and that the terms in (78) are to be taken as in (76), (77), etc., while $R_0^{\frac{1}{n}}$ receives its rational value.

§ 43. By (3), z being any integer,

$$R_{z\sigma}^{\frac{1}{n}} = \{F(w)\}\, R_1^{\frac{z\sigma}{n}},$$

$F(w)$ being a rational function of w. And equation (5) subsists along with (3); that is, w^e being the general primitive n^{th} root of unity,

$$R_{ez\sigma}^{\frac{1}{n}} = \{F(w^e)\}\, R_e^{\frac{z\sigma}{n}}.$$

Taking $z=1$,

$$R_{e\sigma}^{\frac{\lambda^{s-2}}{n}} = B_e R_e^{\frac{\sigma\lambda^{s-2}}{n}},$$

B_e being a rational function of w^e. In like manner, taking $z=\lambda$,

$$R_{e\sigma\lambda}^{\frac{\lambda^{s-3}}{n}} = C_e R_e^{\frac{\sigma\lambda^{s-2}}{n}},$$

C_e being a rational function of w^e. In this way it can be shown that each of the terms in the series

$$R_{e\sigma}^{\frac{\lambda^{s-2}}{n}},\ R_{e\sigma\lambda}^{\frac{\lambda^{s-3}}{n}},\ R_{e\sigma\lambda^2}^{\frac{\lambda^{s-4}}{n}},\ \ldots\ldots,\ R_{e\sigma\lambda^{s-2}}^{\frac{1}{n}},$$

is the product of $R_e^{\frac{\sigma\lambda^{s-2}}{n}}$ by a rational function of w^e. Therefore

$$\left(R_{e\sigma}^{\lambda^{s-2}} R_{e\sigma\lambda}^{\lambda^{s-3}} R_{e\sigma\lambda^2}^{\lambda^{s-4}} \ldots\ldots R_{e\sigma\lambda^{s-2}}\right)^{\frac{\sigma}{n}} = F_e R_e^{\frac{d}{n}} \tag{79}$$

where F_e is a rational function of w^e, and

$$d = \sigma^2 (s-1)\,\lambda^{s-2}. \tag{80}$$

So also

$$\left.\begin{array}{l} \left(R_{e\tau}^{h^{t-2}} R_{e\tau h}^{h^{t-3}} \ldots\ldots R_{e\tau h^{t-2}}\right)^{\frac{\tau}{n}} = G_e R_e^{\frac{\delta}{n}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left(R_{e\beta}^{k^{b-2}} R_{e\beta k}^{k^{b-3}} \ldots\ldots R_{e\beta k^{b-2}}\right)^{\frac{\beta}{n}} = H_e R_e^{\frac{D}{n}} \end{array}\right\} \tag{81}$$

where G_e, H_e, etc., are rational functions of w^e, and

$$\left.\begin{array}{l} \delta = \tau^2(t-1)h^{t-2} \\ \ldots\ldots\ldots\ldots \\ D = \beta^2(b-1)k^{b-2} \end{array}\right\} \tag{82}$$

From (79) and (81),

$$(R_{e\sigma}^{\lambda^{s-2}}\ldots)^{\frac{\sigma}{n}}(R_{e\tau}^{h^{t-2}}\ldots)^{\frac{\tau}{n}}\ldots(R_{e\beta}^{k^{b-2}}\ldots)^{\frac{\beta}{n}} = Q_e R_e^{\frac{\Delta}{n}}, \tag{83}$$

where Q_e is a rational function of w^e, and Δ is the sum of the terms $d, \delta, \ldots, D$; that is, by (80) and (82),

$$\Delta = \sigma^2(s-1)\lambda^{s-2} + \tau^2(t-1)h^{t-2} + \ldots + \beta^2(b-1)k^{b-2}. \tag{84}$$

Because $b\beta = n = s\sigma$, and the prime numbers b and s are factors of n distinct from one another, b is a factor of σ. Hence b is a factor of the first of the separate members of the expression for Δ in (84). In like manner b is a factor of all the separate members of the expression for Δ except the last. And it is not a factor of the last. For, assuming the prime factors of n in (67) to be all odd, since the last line in (69) is a cycle of primitive b^{th} roots of unity, k is prime to b. And $b-1$ is necessarily prime to b. And β is prime to b, because β is the continued product of those prime factors of n which are distinct from b. Hence $\beta^2(b-1)k^{b-2}$ is prime to b. The conclusion still holds if b is not odd, but equal to 2. For, in that case, $k=1$ and $b-1=1$; so that

$$\beta^2(b-1)k^{b-2} = \beta^2.$$

Now, β^2 is odd, because β is the continued product of the odd factors of n. Hence β^2 is prime to b or 2. Whether, therefore, the terms in (67) are all odd or not, every one of the separate members of the expression for Δ in (84) except the last is divisible by b, but the last is not divisible by b. Hence Δ is prime to b. In like manner Δ is prime to each of the factors of n. Therefore it is prime to n. Therefore there are whole numbers m and r such that

$$m\Delta = rn + 1.$$

Therefore, from (83),

$$(R_{e\sigma}^{\lambda^{s-2}}\ldots)^{\frac{m\sigma}{n}}(R_{e\tau}^{h^{t-2}}\ldots)^{\frac{m\tau}{n}}\ldots(R_{e\beta}^{k^{b-2}}\ldots)^{\frac{m\beta}{n}} = (Q_e^m R_e^r)\, R_e^{\frac{1}{n}}.$$

For any integral value of z, let $(R_z^m)^{\frac{1}{n}}$ be written $P_z^{\frac{1}{n}}$. Then, putting A_e^{-1} for $Q_e^m R_e^r$, $R_e^{\frac{1}{n}} A_e^{-1}$ is the continued product of the expressions

$$\begin{array}{l}
(P_{e\sigma}^{\lambda^{s-2}} P_{e\sigma\lambda}^{\lambda^{s-3}} \ldots P_{e\sigma\lambda^{s-2}})^{\frac{\sigma}{n}}, \\
(P_{e\tau}^{h^{t-2}} P_{e\tau h}^{h^{t-3}} \ldots P_{e\tau h^{t-2}})^{\frac{\tau}{n}}, \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
(P_{e\delta}^{l^{d-2}} P_{e\delta l}^{l^{d-3}} \ldots P_{e\delta l^{d-2}})^{\frac{\delta}{n}}, \\
(P_{e\beta}^{k^{b-2}} P_{e\beta k}^{k^{b-3}} \ldots P_{e\beta k^{b-2}})^{\frac{\beta}{n}};
\end{array}$$

therefore, by (70),

$$R_e^{\frac{1}{n}} = A_e(\phi_\sigma^\sigma \psi_\tau^\tau \ldots . X_\delta^\delta F_\beta^\beta)^{\frac{1}{n}}. \tag{85}$$

Therefore

$$R_1 = A_1^n(\phi_\sigma^\sigma \psi_\tau^\tau \ldots . X_\delta^\delta F_\beta^\beta).$$

Thus the form of the fundamental element in (72) is established. Also, it was necessary to take $R_0^{\frac{1}{n}}$ with its rational value, because, by § 5, $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation $f(x) = 0$. And equation (85) is identical with (76), which establishes the necessity of the forms assigned to all those expressions which are contained under $R_e^{\frac{1}{n}}$. It remains to prove that the expressions contained under $R_{ev}^{\frac{1}{n}}$, $\frac{n}{v}$ or y being a term in the series (75) distinct from n, have the forms assigned to them in (77).

§ 44. Since $yv = n$, and y is not equal to n, y is the continued product of some of the prime factors of n, but not of them all. Let s, t, etc., be the factors of n that are factors of y, while b, d, etc., are not factors of y. Because $yv = n = b\beta$, and b is not a factor of y, b is a factor of v. Let $v = ab$; then $v\beta = an$. Therefore $w^{ev\beta} = w^{ean} = w^0$. Therefore $F_{ev\beta} = F_0$. In like manner $X_{ev\delta} = X_0$. And so on as regards all those terms of the type $F_{ev\beta}$ in which $\frac{n}{\beta}$ or b is not a measure of y. Hence, putting ev for z in the second of equations (74), and separating those factors of $R_{ev}^{\frac{1}{n}}$ that are of the type $F_{ev\beta}^{\frac{1}{n}}$ from those that are not,

$$R_{ev}^{\frac{1}{n}} = w'' A_{ev}(F_0^\beta X_0^\delta \ldots .)^{\frac{1}{n}} (\phi_{ev\sigma}^\sigma \psi_{ev\tau}^\tau \ldots .)^{\frac{1}{n}}, \tag{86}$$

w'' being an n^{th} root of unity. We understand that $F_0^{\frac{\beta}{n}}$, $X_0^{\frac{\delta}{n}}$, etc., are here taken with the rational values which it has been proved that they admit. The continued product of these expressions may be called Q, which gives us

$$R_{ev}^{\frac{1}{n}} = w'' A_{ev} Q (\phi_{ev\sigma}^\sigma \psi_{ev\tau}^\tau \ldots .)^{\frac{1}{n}}.$$

When e is taken with the particular value c, let w'' become w^r, and when e has the value unity, let w'' become w^a. Then

and

$$\left.\begin{aligned} R_{cv}^{\frac{1}{n}} &= w^r A_{cv} Q (\phi_{cv\sigma}^\sigma \psi_{cv\tau}^\tau \ldots .)^{\frac{1}{n}} \\ R_v^{\frac{1}{n}} &= w^a A_v Q (\phi_{v\sigma}^\sigma \psi_{v\tau}^\tau \ldots .)^{\frac{1}{n}} \end{aligned}\right\} \tag{87}$$

Because equations (3) and (5) subsist together, and w^c is included under w^e,

and

$$\left.\begin{aligned} R_v^{\frac{1}{n}} &= k_1 R_1^{\frac{v}{n}} \\ R_{cv}^{\frac{1}{n}} &= k_c R_c^{\frac{v}{n}} \end{aligned}\right\} \tag{88}$$

where k_1 is a rational function of w, and k_c is what k_1 becomes by changing w into w^c. By putting $c = 1$ in (85),

$$R_1^{\frac{1}{n}} = A_1(\phi_\sigma^\sigma \psi_\tau^\tau \ldots . X_\delta^\delta F_\beta^\beta)^{\frac{1}{n}}.$$

Taking this in connection with the second of equations (87),

$$(R_v R_1^{-v})^{\frac{1}{n}} = w^a (A_v A_1^{-v}) Q (F_\beta^{-v\beta} X_\delta^{-v\delta} \ldots .)^{\frac{1}{n}} \{(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})(\psi_{v\tau}^\tau \psi_\tau^{-v\tau}) \ldots .\}^{\frac{1}{n}}. \quad (89)$$

In like manner, by putting c for e in (85), and taking the result in connection with the first of equations (87),

$$(R_{cv} R_c^{-v})^{\frac{1}{n}} = w^r (A_{cv} A_c^{-v}) Q (F_{c\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} \{(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})(\psi_{cv\tau}^\tau \psi_{c\tau}^{-v\tau}) \ldots .\}^{\frac{1}{n}}. \quad (90)$$

From (89) compared with the first of equations (88), and from (90) compared with the second of equations (88),

$$\left.\begin{aligned} k_1 &= w^a (A_v A_1^{-v}) Q (F_\beta^{-v\beta} \ldots .)^{\frac{1}{n}} \{(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})(\psi_{v\tau}^\tau \psi_\tau^{-v\tau}) \ldots .\}^{\frac{1}{n}} \\ \text{and} \quad k_c &= w^r (A_{cv} A_c^{-v}) Q (F_{c\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} \{(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})(\psi_{cv\tau}^\tau \psi_{c\tau}^{-v\tau}) \ldots .\}^{\frac{1}{n}} \end{aligned}\right\} \quad (91)$$

By § 9, because ϕ_σ is of the same structure as the expression (8),

$$(\phi_{v\sigma} \phi_\sigma^{-v})^{\frac{1}{s}} = q_\sigma,$$

q_σ being a rational function of the primitive s^{th} root of unity w^σ. And, since it appeared from the reasoning in § 9 that the nature of the function does not depend on the particular primitive s^{th} root of unity denoted by w^σ, we have at the same time

$$(\phi_{cv\sigma} \phi_{c\sigma}^{-v})^{\frac{1}{s}} = q_{c\sigma},$$

$q_{c\sigma}$ being what q_σ becomes when w is changed into w^c. Therefore, because $s\sigma = n$,

$$(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})^{\frac{1}{n}} = q_\sigma$$

and

$$(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})^{\frac{1}{n}} = q_{c\sigma}.$$

Similarly,

$$(\psi_{v\tau}^\tau \psi_\tau^{-v\tau})^{\frac{1}{n}} = q'_\tau$$

and

$$(\psi_{cv\tau}^\tau \psi_{c\tau}^{-v\tau})^{\frac{1}{n}} = q'_{c\tau},$$

where q'_τ is a rational function of w^τ, and $q'_{c\tau}$ is what q'_τ becomes when w is changed into w^c. Therefore, from (91),

$$\left.\begin{aligned} k_1 &= w^a (A_v A_1^{-v}) Q (F_\beta^{-v\beta} \ldots .)^{\frac{1}{n}} (q_\sigma q'_\tau \ \ldots .) \\ \text{and} \quad k_c &= w^r (A_{cv} A_c^{-v}) Q (F_{c\beta}^{-v\beta} \ldots .)^{\frac{1}{n}} (q_{c\sigma} q'_{c\tau} \ldots .) \end{aligned}\right\}. \quad (92)$$

But again, because $b\beta = n = yv$, and y is not a multiple of b, v is a multiple of b. Therefore $v\beta$ is a multiple of $b\beta$ or n. Therefore $F_{c\beta}^{\frac{-v\beta}{n}}$ is a rational

function of w^e. In like manner $X_{e\delta}^{\frac{-v\delta}{n}}$ is a rational function of w^e, and so on. Therefore the second of equations (92) may be written

$$k_c = w^r (A_{cv} A_c^{-v})\, Q M_c (q_{cv} q'_{cr} \dots),$$

where M_c is a rational function of w^e. In like manner, from the first of equations (92),

$$k_1 = w^a (A_v A_1^{-v})\, Q M_1 (q_v q'_r \dots),$$

M_1 being what M_c becomes in passing from w^e to w. By § 4 we can change w in this last equation into w^e. This gives us

$$k_c = w^{ac} (A_{cv} A_c^{-v})\, Q M_c (q_{cv} q'_{cr} \dots).$$

Comparing this with the value of k_c previously obtained, $w^r = w^{ac}$. Therefore the first of equations (87) becomes

$$R_{cv}^{\frac{1}{n}} = w^{ac} A_{cv} Q (\phi_{cv\sigma}^{\sigma} \psi_{cv\tau}^{\tau} \dots)^{\frac{1}{n}}.$$

Replacing Q by $(F_{cv\beta}^{\beta} X_{cv\delta}^{\delta} \dots)^{\frac{1}{n}}$, and putting e for c, which we are entitled to do because w^e may be any one of the roots included under w^e,

$$R_{ev}^{\frac{1}{n}} = w^{ea} A_{ev} (\phi_{ev\sigma}^{\sigma} \psi_{ev\tau}^{\tau} \dots F_{ev\beta}^{\beta})^{\frac{1}{n}},$$

which is the form of $R_{ev}^{\frac{1}{n}}$ in (77).

Sufficiency of the Forms.

§ 45. Here we assume that R_1 has the form (72), and that the terms in (78) are determined by the equations (76), (77), etc., while $R_0^{\frac{1}{n}}$ receives its rational value. We have then to prove that the expression (73) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible.

§ 46. *In the first place*, it has been shown that there is an n^{th} root of R_0 which has a rational value; and, by hypothesis, $R_0^{\frac{1}{n}}$ has been taken with this rational value. *In the second place*, an equation of the type (3) subsists for every integral value of z. For, let z be a multiple of n. In that case it may be taken to be zero. Then

$$(R_z R_1^{-z})^{\frac{1}{n}} = R_0^{\frac{1}{n}}. \tag{93}$$

But R_0 is the n^{th} power of a rational quantity. Therefore (93) is an equation of the type (3). If z is not a multiple of n, it may be a multiple of some of the factors of n, say b, d, etc., though not of others, say s, t, etc. Because z is a multiple of b, and $b\beta = n$, $z\beta$ is a multiple of n. Therefore $F_{z\beta} = F_0$. And F_0^{β} is the n^{th} power of a rational quantity. Therefore $F_{z\beta}^{\beta}$ is the n^{th} power of a

rational quantity. In like manner $X_{z\delta}^{\delta}$ is the n^{th} power of a rational quantity, and so on. But

$$R_z = A_z^n (F_{z\beta}^{\beta} X_{z\delta}^{\delta} \dots)(\phi_{z\sigma}^{\sigma} \psi_{z\tau}^{\tau} \dots).$$

Since each of the quantities $F_{z\beta}^{\beta}$, $X_{z\delta}^{\delta}$, etc., is the n^{th} power of a rational quantity, let their continued product be Q^n, Q being rational. Then

$$R_z = (A_z Q)^n (\phi_{z\sigma}^{\sigma} \psi_{z\tau}^{\tau} \dots). \tag{94}$$

Again, because $z\beta$ is a multiple of n, $F_{\beta}^{-z\beta}$ is the n^{th} power of a rational function of w. In like manner $X_{\delta}^{-z\delta}$ is the n^{th} power of a rational function of w, and so on. Let

$$(F_{\beta}^{-z\beta} X_{\delta}^{-z\delta} \dots) = M_1^{-n},$$

M_1 being a rational function of w. Then

$$\begin{aligned} R_1^{-z} &= A_1^{-nz} (F_{\beta}^{-\beta z} \dots)(\phi_{\sigma}^{-\sigma z} \psi_{\tau}^{-\tau z} \dots) \\ &= (A_1^z M_1)^{-n} (\phi_{\sigma}^{-\sigma z} \psi_{\tau}^{-\tau z} \dots). \end{aligned} \tag{95}$$

From (94) and (95),

$$R_z R_1^{-z} = (A_z A_1^{-z})^n (Q M_1^{-1})^n \{(\phi_{z\sigma}^{\sigma} \phi_{\sigma}^{-z\sigma})(\psi_{z\tau}^{\tau} \psi_{\tau}^{-z\tau}) \dots\}. \tag{96}$$

From the structure of the expression ϕ_σ, $\phi_{z\sigma}\phi_{\sigma}^{-z}$ is, by § 9, the s^{th} power of a rational function of w^σ. Therefore, because $s\sigma = n$, $\phi_{z\sigma}^{\sigma}\phi_{\sigma}^{-z\sigma}$ is the n^{th} power of a rational function of w. In like manner $\psi_{z\tau}^{\tau}\psi_{\tau}^{-z\tau}$ is the n^{th} power of a rational function of w, and so on. Therefore, from (96), $R_z R_1^{-z}$ is the n^{th} power of a rational function of w. This establishes equation (3) when z is the continued product of some of the prime factors of n, but not of all. It virtually establishes equation (3) also when z is prime to n, because this case may be regarded as included in the preceding by taking the view that the factors of n which measure z have disappeared. Thus, whether z be a multiple of n or be a multiple of some factors of n, but not of others, or be prime to n, an equation of the type (3) subsists. *In the third place*, an equation of the type (5) subsists along with (3) for every value of e that makes w^e a primitive n^{th} root of unity. For, let z be prime to n. It is then included in e. Also, since z and e are both prime to n, ze is included in e; and unity is included in e. But, from the manner in which the root was constructed from its fundamental element, $R_e^{\frac{1}{n}}$ is determined as in (76). Therefore we have the four equations

$$\begin{aligned} R_1^{\frac{1}{n}} &= A_1 (\phi_{\sigma}^{\sigma} \psi_{\tau}^{\tau} \quad \dots \, F_{\beta}^{\beta})^{\frac{1}{n}}, \\ R_z^{\frac{1}{n}} &= A_z (\phi_{z\sigma}^{\sigma} \psi_{z\tau}^{\tau} \quad \dots \, F_{z\beta}^{\beta})^{\frac{1}{n}}, \\ R_{ez}^{\frac{1}{n}} &= A_{ez} (\phi_{ez\sigma}^{\sigma} \psi_{ez\tau}^{\tau} \quad \dots \, F_{ez\beta}^{\beta})^{\frac{1}{n}}, \\ R_e^{\frac{1}{n}} &= A_e (\phi_{e\sigma}^{\sigma} \psi_{e\tau}^{\tau} \quad \dots \, F_{e\beta}^{\beta})^{\frac{1}{n}}. \end{aligned}$$

Therefore and

$$\left.\begin{aligned}(R_z R_1^{-z})^{\frac{1}{n}} &= (A_z A_1^{-z})(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{n}} \ldots\ldots (F_{z\beta}F_\beta^{-z})^{\frac{\beta}{n}}; \\ (R_{ez} R_e^{-z})^{\frac{1}{n}} &= (A_{ez} A_e^{-z})(\phi_{ez\sigma}\phi_{e\sigma}^{-z})^{\frac{\sigma}{n}} \ldots\ldots (F_{ez\beta}F_{e\beta}^{-z})^{\frac{\beta}{n}}\end{aligned}\right\} \tag{97}$$

Because $(\phi_{z\sigma}\phi_\sigma^{-z})^{\frac{\sigma}{n}}$ and other corresponding expressions have been shown to be rational functions of the primitive n^{th} root of unity w, the two equations (97) correspond respectively to (3) and (5). If z be not prime to n, and yet not a multiple of n, it may be taken to be ev, where v is equal to $\frac{n}{y}$, y being one of the terms in the series (75) distinct from n, and w^e being the general primitive n^{th} root of unity. Then, just as we obtained the pair of equations (97) by means of (76), we can now, by means of (77), obtain the pair of equations

$$\left.\begin{aligned}(R_{ev} R_1^{-ev})^{\frac{1}{n}} &= (A_{ev} A_1^{-ev})(\phi_{ev\sigma}\phi_\sigma^{-ev})^{\frac{\sigma}{n}} \ldots\ldots \\ (R_{cev} R_c^{-ev})^{\frac{1}{n}} &= (A_{cev} A_c^{-ev})(\phi_{cev\sigma}\phi_{c\sigma}^{-ev})^{\frac{\sigma}{n}} \ldots\ldots\end{aligned}\right\} \tag{98}$$

where w^c represents any one of the primitive n^{th} roots of unity. Because such expressions as $(\phi_{ev\sigma}\phi_\sigma^{-ev})^{\frac{\sigma}{n}}$ and $(\phi_{cev\sigma}\phi_{c\sigma}^{-ev})^{\frac{\sigma}{n}}$ are rational functions of w, the two equations (98) correspond respectively to (3) and (5). Finally, should z be a multiple of n, it may be taken to be zero. Then the equation corresponding to (3) is, q_1 being a rational function of w,

$$R_z^{\frac{1}{n}} = q_1 R_1^{\frac{z}{n}}; \text{ or, since } z = 0,\ R_0^{\frac{1}{n}} = q_1.$$

But $R_0^{\frac{1}{n}}$ is rational. Therefore q_1 is rational. Therefore $q_1 = q_e$; in other words, q_1 undergoes no change when w becomes w^e. Also $R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}} = q_e$. Therefore, since $R_e^{\frac{z}{n}} = 1$,

$$R_{ez}^{\frac{1}{n}} = q_e R_e^{\frac{z}{n}},$$

which is the equation corresponding to (5). Therefore, whatever z be, the equation (5) subsists along with (3). Hence, by the Criterion in § 10, the expression (73) is the root of a pure uni-serial Abelian equation of the n^{th} degree.

The Pure Uni-Serial Abelian of a Degree which is Four Times the Continued Product of a Number of Distinct Odd Primes.

Fundamental Element of the Root.

§ 47. Let $n = 4m$, where m is the continued product of the distinct odd prime numbers,

$$s,\ t,\ \ldots\ldots,\ d,\ b. \tag{99}$$

Take

$$\sigma,\ \tau,\ \ldots\ldots,\ \delta,\ \beta, \tag{100}$$

such that $n = s\sigma = t\tau = \ldots\ldots = b\beta$. Let w be a primitive n^{th} root of unity. Then w^m is a primitive fourth root of unity, w^σ a primitive s^{th} root of unity, and so on. Let

$$\left.\begin{array}{l} w^\sigma,\ w^{\sigma\lambda},\ w^{\sigma\lambda^2}, \ldots\ldots,\ w^{\sigma\lambda^{s-2}} \\ w^\tau,\ w^{\tau h},\ w^{\tau h^2}, \ldots\ldots,\ w^{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ w^\delta,\ w^{\delta l},\ w^{\delta l^2}, \ldots\ldots,\ w^{\delta l^{d-2}} \\ w^\beta,\ w^{\beta k},\ w^{\beta k^2}, \ldots\ldots,\ w^{\beta k^{b-2}} \end{array}\right\} \qquad (101)$$

be cycles containing respectively all the primitive s^{th} roots of unity, all the primitive t^{th} roots of unity, and so on. Let P_1 be a rational function of w, and, for any integral value of z, let P_z be what P_1 becomes by changing w into w^z. We can always take P_1 such that P_m shall have the form of the fundamental element of the root of a pure uni-serial Abelian quartic; that is, P_m may receive the form of R_1 in (49) as determined by the equations (48). For, because P_1 is a rational function of w,

$$P_1 = a + a_1 w + a_2 w^2 + \ldots\ldots + a_{n-1} w^{n-1},$$

the coefficients a, a_1, etc., being rational. Therefore

$$\begin{aligned} P_m &= a + a_1 w^m + a_2 w^{2m} + \text{etc.} \\ &= (a + a_4 + \text{etc.}) + w^m (a_1 + a_5 + \text{etc.}) + w^{2m} (a_2 + \text{etc.}) + w^{3m} (a_3 + \text{etc.}). \end{aligned}$$

This may be written

$$P_m = f + f' w^m + f'' w^{2m} + f''' w^{3m}. \qquad (102)$$

All that is required in order that P_1 may be a function of the kind described is that P_m in (102) be of the same character with R_1 in (49). That is, we have to make

$$f = p,\ f' = q,\ f'' = r,\ f''' = s.$$

By means of these four linear equations, the necessary relations between the quantities a, a_1, a_2, etc., can be constituted. Having thus taken P_1 subject to the condition that P_m shall have the form of the fundamental element of the root of a pure uni-serial Abelian quartic, put

$$\left.\begin{array}{l} \Phi_\sigma = P_\sigma^{\lambda^{s-2}} P_{\sigma\lambda}^{\lambda^{s-3}} P_{\sigma\lambda^2}^{\lambda^{s-4}} \ldots\ldots P_{\sigma\lambda^{s-2}} \\ \psi_\tau = P_\tau^{h^{t-2}} P_{\tau h}^{h^{t-3}} P_{\tau h^2}^{h^{t-4}} \ldots\ldots P_{\tau h^{t-2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_\delta = P_\delta^{l^{d-2}} P_{\delta l}^{l^{d-3}} P_{\delta l^2}^{l^{d-4}} \ldots\ldots P_{\delta l^{d-2}} \\ F_\beta = P_\beta^{k^{b-2}} P_{\beta k}^{k^{b-3}} P_{\beta k^2}^{k^{b-4}} \ldots\ldots P_{\beta k^{b-2}} \end{array}\right\} \qquad (103)$$

Then, if R_1 be the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree, it will be found that

$$R_1 = A_1^n (P_m^m \phi_\sigma^\sigma \psi_\tau^\tau \ldots . X_\delta^\delta F_\beta^\beta), \tag{104}$$

A_1 being a rational function of w.

The Root Constructed from its Fundamental Element.

§ 48. From R_1, as expressed in (104), derive R_0, R_2, etc., by changing w into w^0, w^2, etc. Then, assuming that the root of the pure uni-serial Abelian equation $f(x) = 0$ of the n^{th} degree is

$$R_0^{\frac{1}{n}} + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots . + R_{n-1}^{\frac{1}{n}}, \tag{105}$$

what we have to do in order to construct the root is to determine what values of $R_0^{\frac{1}{n}}$, $R_1^{\frac{1}{n}}$, etc., are to be taken together in (105).

§ 49. From (104) we have

$$R_0 = A_0^n (P_0^m \phi_0^\sigma \ldots . F_0^\beta). \tag{106}$$

By § 8, ϕ_0 is the s^{th} power of a rational quantity. Therefore, because $s\sigma = n$, ϕ_0^σ is the n^{th} power of a rational quantity. In like manner each of the expressions ψ_0^τ, F_0^β, etc., is the n^{th} power of a rational quantity. And, because P_m is of the same form with the fundamental element of the root of a pure uni-serial Abelian quartic, P_0 is the fourth power of a rational quantity. Therefore, since $n = 4m$, P_0^m is the n^{th} power of a rational quantity. Therefore, from (106), R_0 is the n^{th} power of a rational quantity, and $R_0^{\frac{1}{n}}$ has a rational value.

§ 50. Let the numbers not exceeding n that measure n, unity not included, be

$$n,\ y,\ \text{etc.} \tag{107}$$

For instance, if $n = 4 \times 3 \times 5 = 60$, the series (107) is

$$60,\ 30,\ 20,\ 15,\ 12,\ 10,\ 6,\ 5,\ 4,\ 3,\ 2.$$

The n^{th} roots of unity distinct from unity are made up of the primitive n^{th} roots of unity, the primitive y^{th} roots of unity, and so on. For instance, when $n = 60$, the fifty-nine n^{th} roots of unity distinct from unity are the sixteen primitive 60^{th} roots of unity, and the eight primitive 30^{th} roots of unity, and the eight primitive 20^{th} roots of unity, and the eight primitive 15^{th} roots of unity, and the four primitive 12^{th} roots of unity, and the four primitive 10^{th} roots of unity, and the two primitive 6^{th} roots of unity, and the four primitive 5^{th} roots of unity, and the two primitive 4^{th} roots of unity, and the two primitive 3^{d} roots of unity, and

the primitive 2^d root of unity. According to our usual notation, let P_z, ϕ_z, etc., be what P_1, ϕ_1, etc., become when w is changed into w^z, z being any integer. Then, from (104),

$$\left.\begin{aligned} R_z &= A_z^n (P_{zm}^m \phi_{z\sigma}^\sigma \psi_{z\tau}^\tau \ldots \ldots F_{z\beta}^\beta) \\ \text{Therefore} \quad R_z^{\frac{1}{n}} &= w' A_z (P_{zm}^m \phi_{z\sigma}^\sigma \psi_{z\tau}^\tau \ldots \ldots F_{z\beta}^\beta)^{\frac{1}{n}} \end{aligned}\right\} \tag{108}$$

w' being an n^{th} root of unity. The general primitive n^{th} root of unity being w^e, give w' in the second of equations (108) the value unity for every value of z included under e. Then

$$R_e^{\frac{1}{n}} = A_e (P_{em}^m \phi_{e\sigma}^\sigma \psi_{e\tau}^\tau \ldots \ldots F_{e\beta}^\beta)^{\frac{1}{n}}. \tag{109}$$

Taking any number y distinct from n in the series (107), since y is a factor of n, let $yv = n$. Then w^v is a primitive y^{th} root of unity. Hence, since w^e is the general primitive n^{th} root of unity, all the primitive y^{th} roots of unity are included in w^{ev}. If w' in the second of equations (108) be w^a when $z = v$, give w' the value w^{ea} when $z = ev$. Then

$$R_{ev}^{\frac{1}{n}} = w^{ea} A_{ev} (P_{evm}^m \phi_{ev\sigma}^\sigma \ldots \ldots F_{ev\beta}^\beta)^{\frac{1}{n}}. \tag{110}$$

The expression P_m having the form of the fundamental element of the root of a pure uni-serial Abelian quartic, it is understood that, in (110), $P_{evm}^{\frac{m}{n}}$ or $P_{evm}^{\frac{1}{4}}$ is taken with the value which it has in the root

$$P_0^{\frac{1}{4}} + P_m^{\frac{1}{4}} + P_{2m}^{\frac{1}{4}} + P_{3m}^{\frac{1}{4}}$$

of a pure uni-serial Abelian quartic; and consequently, when v is a multiple of 2, w^{ma} must have the value unity. Form equations similar to (110) for the remaining terms in (107). In this way, because the series of the n^{th} roots of unity distinct from unity is made up of the primitive n^{th} roots of unity, and the primitive y^{th} roots of unity, and so on, all the terms $1, 2, \ldots, n-1$ will be found in the groups of numbers represented by the subscripts e, ev, etc., when multiples of n are rejected. Consequently, in determining $R_e^{\frac{1}{n}}$, $R_{ev}^{\frac{1}{n}}$, etc., as in (109), (110), etc., we have determined all the terms

$$R_1^{\frac{1}{n}},\ R_2^{\frac{1}{n}},\ \ldots,\ R_{n-1}^{\frac{1}{n}}. \tag{111}$$

Substitute, then, in (105) the rational value of $R_0^{\frac{1}{n}}$, and the terms in (111) as these are determined by the equations (109), (110), etc., and the root is constructed; that is, the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible.

Necessity of the above Forms.

§51. Take s, any one of the odd prime numbers in the series (99). Let a_0, a_1, a_2, etc., be rational functions of w^s. Then, because $s\sigma = n$, a_0, a_1, etc., are clear of w^σ, though they may involve the primitive fourth root of unity w^m, the primitive t^{th} root of unity w^τ, and other corresponding roots exclusive of w^σ. The terms w^σ, $w^{\sigma\lambda}$, etc., in the first of the cycles (101), being all the primitive s^{th} roots of unity, I assume that if

$$a_0 + a_1 w^\sigma + a_2 w^{\sigma\lambda} + \ldots + a_{s-1} w^{\sigma\lambda^{s-2}} = 0,$$

the coefficients a_0, a_1, etc., are all equal to one another.

§52. The general primitive n^{th} root of unity being w^e, $s-1$ values of e, leaving distinct residues when multiples of s are rejected, can be found of the form

$$g\sigma + 1, \tag{112}$$

g being a whole number. For, since $s\sigma = n$, the $s-1$ terms

$$\sigma + 1,\ 2\sigma + 1,\ \ldots,\ (s-1)\sigma + 1 \tag{113}$$

are all less than n. Of these terms, not more than one can have a measure in common with n. For suppose, if possible, that two of the terms in (113), $a\sigma + 1$ and $b\sigma + 1$, have a measure in common with n. The measure which $a\sigma + 1$ has in common with n cannot be any of the measures of σ. Therefore, since $s\sigma = n$, it must be the prime number s. We may therefore put

$$a\sigma + 1 = hs.$$

In like manner,

$$b\sigma + 1 = ks,$$

h and k being whole numbers. Therefore, assuming $a - b$ to be positive,

$$(a-b)\sigma = (h-k)s.$$

But $a - b$ is less than the prime number s. It is therefore a measure of $h - k$. Therefore σ is a multiple of s; which, because σ is four times the continued product of the odd prime factors of n exclusive of s, is impossible. Hence not more than one of the $s-1$ terms in (113) can have a measure in common with n. In other words, $s-2$ of the terms in (113) are prime to n. Therefore $s-1$ of the roots

$$w,\ w^{\sigma+1},\ w^{2\sigma+1},\ \ldots,\ w^{(s-1)\sigma+1}$$

are primitive n^{th} roots of unity. This implies that there are $s-1$ values of g

in (112), zero included, which make $w^{g\sigma+1}$ a primitive n^{th} root of unity. Let two of these values of g be g_1 and g_2. Put

$$g_1\sigma + 1 = q_1 s + r_1$$

and

$$g_2\sigma + 1 = q_2 s + r_2,$$

q_1 and q_2 being whole numbers, and r_1 and r_2 whole numbers less than s. Suppose, if possible, that $r_1 = r_2$; then

$$(g_1 - g_2)\sigma = (q_1 - q_2)s,$$

which, as above, makes σ a multiple of s, and is therefore impossible. Consequently, the $s-1$ residues after multiples of s have been rejected from the $s-1$ different values of $g\sigma + 1$ are all different from one another.

§ 53. It can now be shown that equations

$$\left.\begin{aligned} (R_{mz}R_m^{-z})^{\frac{1}{s}} &= p_m \\ \text{and} \quad (R_{emz}R_{em}^{-z})^{\frac{1}{s}} &= p_{em} \end{aligned}\right\} \tag{114}$$

subsist for every integral value of z and every value of e that makes w^e a primitive n^{th} root of unity, p_m being a rational function of w^m, and p_{em} being what p_m becomes when w is changed into w^e. By (3) and (5), because R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree,

$$(R_{mz}R_m^{-z})^{\frac{1}{n}} = k_1,$$

and

$$(R_{emz}R_{em}^{-z})^{\frac{1}{n}} = k_e,$$

k_1 being a rational function of w, and k_e being what k_1 becomes when w is changed into w^e. Therefore

$$\left.\begin{aligned} (R_{mz}R_m^{-z})^{\frac{1}{s}} &= k_1^m \\ \text{and} \quad (R_{emz}R_{em}^{-z})^{\frac{1}{s}} &= k_e^m \end{aligned}\right\} \tag{115}$$

In the second of these equations, give e a value, say c, falling under the form (112). Then

$$(R_{cmz}R_{cm}^{-z})^{\frac{1}{s}} = k_c^m. \tag{116}$$

Since σ is a multiple of 4, we may put $c = 4d + 1$. Therefore $cm = dn + m$. Therefore $w^{cm} = w^m$, and $w^{cmz} = w^{mz}$. Therefore (116) may be written

$$(R_{mz}R_m^{-z})^{\frac{1}{s}} = k_c^m.$$

This, compared with the first of equations (115), gives us

$$k_c^m = k_1^m. \tag{117}$$

Since k_1^m is a rational function of a primitive n^{th} root of unity, and the first of the cycles (101) contains all the primitive s^{th} roots of unity, we may put

$$k_1^m = a_0 + a_1 w^\sigma + a_2 w^{\sigma\lambda} + \ldots + a_{s-1} w^{\sigma\lambda^{s-2}}, \tag{118}$$

where the coefficients a_0, a_1, etc., are clear of w^σ; though, for anything that has yet been proved, they may involve w^m, w^τ and other corresponding roots exclusive of w^σ. In (118), by the Corollary in § 4, we can change w into w^c. This causes k_1^m to become k_c^m, and w^σ to become $w^{c\sigma}$. The coefficients a_0, a_1, etc., are rational functions of w^s, and, when w is changed into w^c, w^s becomes w^{cs}; but, by (112), $cs = gn + s$; therefore $w^{cs} = w^s$. This implies that the coefficients a_0, a_1, etc., remain unaffected when w is changed into w^c. Therefore

$$k_c^m = a_0 + a_1 w^{c\sigma} + a_2 w^{c\sigma\lambda} + \text{etc.}$$

Therefore, from (117) and (118),

$$a_1 w^{c\sigma} + a_2 w^{c\sigma\lambda} + \text{etc.} = a_1 w^\sigma + a_2 w^{\sigma\lambda} + \text{etc.} \tag{119}$$

It was proved in § 52 that c may have $s - 1$ values, including unity, which leave distinct residues when multiples of s are rejected. Therefore one of these residues distinct from unity must be λ, which was supposed less than s, and is not unity. Giving c in (119) the value which leaves the residue λ when multiples of s are rejected, the equation (119) becomes

$$w^{\sigma\lambda}(a_1 - a_2) + w^{\sigma\lambda^2}(a_2 - a_3) + \text{etc.} = 0.$$

Here, by § 51, the coefficients $a_1 - a_2$, $a_2 - a_3$, etc., must all vanish. This implies that $a_1, a_2, \ldots\ldots, a_{s-1}$ are all equal to one another. Hence

$$k_1^m = a_0 + a_1(w^\sigma + w^{\sigma\lambda} + \text{etc.}) = a_0 - a_1. \tag{120}$$

Thus k_1^m is clear of w^σ. In like manner it can be shown to be clear of all the roots

$$w^\sigma,\ w^\tau,\ \ldots\ldots,\ w^\delta,\ w^\beta;$$

it is therefore a rational function of w^m. Let it be written p_m. Then the equations (115) become

$$(R_{mz} R_m^{-z})^{\frac{1}{4}} = p_m,$$
$$(R_{emz} R_{em}^{-z})^{\frac{1}{4}} = p_{em},$$

p_{em} being what p_m becomes when w is changed into w^e. These are the equations (114).

§ 54. From what has been established, it follows that R_m has the form of the fundamental element of a pure uni-serial Abelian quartic. For, by § 10, all that is required in order that R_m may have such a form is that the equations (114) should subsist, and that $R_0^{\frac{1}{4}}$ should have a rational value. By § 5, since R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree, $R_0^{\frac{1}{n}}$ has a rational value. Therefore $R_0^{\frac{1}{4}}$ has a rational value.

§ 55. In the very same way in which (83) was established, it can be proved that

$$R_{em}^{\frac{m}{n}}(R_{e\sigma}^{\lambda^{s-2}} R_{e\sigma\lambda}^{\lambda^{s-3}} \dots R_{e\sigma\lambda^{s-2}})^{\frac{\sigma}{n}} \dots (R_{e\beta}^{k^{b-2}} \dots)^{\frac{\beta}{n}} = Q_e R_e^{\frac{\Delta}{n}}, \tag{121}$$

where Q_e is a rational function of w^e, and

$$\Delta = m^2 + \sigma^2 (s-1) \lambda^{s-2} + \tau^2 (t-1) h^{t-2} + \dots + \beta^2 (b-1) k^{b-2}. \tag{122}$$

Because m is the continued product of the odd factors of n, m^2 is odd. But each of the expressions $s-1$, $t-1$, etc., is even. Therefore Δ is odd. Therefore Δ is prime to 4. Again, because m is the continued product of the odd factors of n, it is a multiple of b. And, because $s\sigma = b\beta$, σ is a multiple of b. In like manner τ is a multiple of b. In this way all the separate members of the expression for Δ in (122) except the last are multiples of b. And, by the same reasoning as was used in § 44, $\beta^2 (b-1) k^{b-2}$ is not a multiple of b. Therefore Δ is prime to b. In like manner it is prime to s, t, etc. Therefore it is prime to n. Therefore there are whole numbers v and r such that

$$v\Delta = rn + 1.$$

Therefore, from (121),

$$R_{em}^{\frac{mv}{n}} (R_{e\sigma}^{\lambda^{s-2}} \dots)^{\frac{\sigma v}{n}} (R_{e\tau}^{h^{t-2}} \dots)^{\frac{\tau v}{n}} \dots (R_{e\beta}^{k^{b-2}} \dots)^{\frac{\beta v}{n}} = (Q_e^v R_e^r) R_e^{\frac{1}{n}}. \tag{123}$$

For any integral value of z, let $R_z^{\frac{v}{n}}$ be written $P_z^{\frac{1}{n}}$. Then, by (103), putting A_e^{-1} for $Q_e^v R_e^r$, (123) becomes

$$R_e^{\frac{1}{n}} = A_e (P_{em}^m \phi_{e\sigma}^\sigma \psi_{e\tau}^\tau \dots F_{e\beta}^\beta)^{\frac{1}{n}}. \tag{124}$$

Therefore

$$R_1 = A_1^n (P_m^m \phi_\sigma^\sigma \psi_\tau^\tau \dots F_\beta^\beta). \tag{125}$$

But P_m is the same as R_m^v. Therefore, by § 54, P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore the expression for R_1 in (125) is identical with that in (104), and thus the form of the fundamental element in (104) is established. Also, it was necessary to take $R_0^{\frac{1}{n}}$ with its rational value, because, by § 5, $nR_0^{\frac{1}{n}}$ is the sum of the roots of the equation $f(x) = 0$. And equation (124) is identical with (109), which establishes the necessity of the forms assigned to all those expressions which are contained under $R_e^{\frac{1}{n}}$. It remains to prove that the expressions contained under $R_{ev}^{\frac{1}{n}}$, $\frac{n}{v}$ or y being a term in the series (107) distinct from n, have the forms assigned to them in (110). The details to be given here are very much a repetition of what is found in § 44; but, to prevent the confusion that might arise

from explanations and re^c ..nces, it is thought better to present the reasoning again with some fulness.

§ 56. Since $yv = n$, and y is not equal to n, y is the continued product of some of the factors of n, but not of them all. Let s, t, etc., be the odd factors of n of which y is a multiple; and b, d, etc., the odd factors of n of which y is not a multiple. Because $yv = n = b\beta$, and b is not a factor of y, b is a factor of v. Let $v = ab$; then $v\beta = an$. Therefore $F_{ev\beta} = F_0$. In like manner $X_{ev\delta} = X_0$, and so on as regards all those terms of the type $F_{ev\beta}$ in which $\frac{n}{\beta}$ or b is an odd factor of n, but not a factor of y. Hence, putting ev for z in the second of equations (108), and separating those factors of $R_{ev}^{\frac{1}{n}}$ that are of the type $F_{ev\beta}^{\frac{1}{n}}$ from those that are not,

$$R_{ev}^{\frac{1}{n}} = w'' A_{ev} (F_0^{\beta} X_0^{\delta} \ldots .)^{\frac{1}{n}} (P_{evm}^{m} \phi_{ev\sigma}^{\sigma} \ldots .)^{\frac{1}{n}},$$

w'' being an n^{th} root of unity. We understand that $F_0^{\frac{\beta}{n}}$, $X_0^{\frac{\delta}{n}}$, etc., are taken with the rational values which it has been proved that they admit, and, as in § 44, their continued product may be called Q. Then

$$R_{ev}^{\frac{1}{n}} = w'' A_{ev} Q (P_{evm}^{m} \phi_{ev\sigma}^{\sigma} \ldots .)^{\frac{1}{n}}. \tag{126}$$

When e is taken with the particular value c, let w'' become w^r, and when e has the value unity, let w'' become w^a. Then

$$\left.\begin{aligned} R_{cv}^{\frac{1}{n}} &= w^r A_{cv} Q (P_{cvm}^{m} \phi_{cv\sigma}^{\sigma} \ldots .)^{\frac{1}{n}} \\ \text{and} \quad R_{v}^{\frac{1}{n}} &= w^a A_{v} Q (P_{vm}^{m} \phi_{v\sigma}^{\sigma} \ \ldots .)^{\frac{1}{n}} \end{aligned}\right\} \tag{127}$$

Because R_1 is the fundamental element of the root of a pure uni-serial Abelian equation of the n^{th} degree, equations (3) and (5) subsist together; hence, because w^c is included in w^e,

$$\left.\begin{aligned} (R_v R_1^{-v})^{\frac{1}{n}} &= k_1 \\ \text{and} \quad (R_{cv} R_c^{-v})^{\frac{1}{n}} &= k_c \end{aligned}\right\} \tag{128}$$

where k_1 is a rational function of w, and k_c is what k_1 becomes by changing w into w^c. By putting e equal to unity in (109),

$$R_1^{\frac{1}{n}} = A_1 (P_m^{m} \phi_\sigma^{\sigma} \ldots . F_\beta^{\beta})^{\frac{1}{n}}.$$

Taking this in connection with the second of equations (127),

$$(R_v R_1^{-v})^{\frac{1}{n}} = w^a (A_v A_1^{-v}) Q (F_\beta^{-v\beta} \ldots .)^{\frac{1}{n}} \{(P_{vm}^{m} P_m^{-vm})(\phi_{v\sigma}^{\sigma} \phi_\sigma^{-v\sigma}) \ldots .\}^{\frac{1}{n}}. \tag{129}$$

In like manner, by putting c for e in (109), and taking the result in connection with the first of equations (127),

$$(R_{cv}R_c^{-v})^{\frac{1}{n}} = w^r(A_{cv}A_c^{-v})Q(F_{c\beta}^{-v\beta}\ldots)^{\frac{1}{n}}\{(P_{cvm}^m P_{cm}^{-vm})(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})\ldots\}^{\frac{1}{n}}. \tag{130}$$

From (129) compared with the first of equations (128), and from (130) compared with the second of equations (128),

$$\left.\begin{aligned} k_1 &= w^a(A_v A_1^{-v})Q(F_\beta^{-v\beta}\ldots)^{\frac{1}{n}}\{(P_{vm}^m P_m^{-vm})(\phi_{v\sigma}^\sigma \phi_\sigma^{-v\sigma})\ldots\}^{\frac{1}{n}} \\ \text{and}\qquad k_c &= w^r(A_{cv}A_c^{-v})Q(F_{c\beta}^{-v\beta}\ldots)^{\frac{1}{n}}\{(P_{cvm}^m P_{cm}^{-vm})(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})\ldots\}^{\frac{1}{n}} \end{aligned}\right\} \tag{131}$$

Exactly as in § 44, it can be shown that

$$(\phi_{cv\sigma}^\sigma \phi_{c\sigma}^{-v\sigma})^{\frac{1}{n}} = q_c, \tag{132}$$

q_c being a rational function of the primitive n^{th} root of unity w^c. Also, it has been proved that P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore, by (3), $(P_{cvm}P_{cm}^{-v})^{\frac{1}{4}}$ is a rational function of the primitive fourth root of unity w^m. Therefore, because $n = 4m$, $(P_{cvm}^m P_{cm}^{-vm})^{\frac{1}{n}}$ is a rational function of the primitive n^{th} root of unity w^c. Put

$$(P_{cvm}^m P_{cm}^{-vm})^{\frac{1}{n}} = q_c'. \tag{133}$$

Again, exactly as in § 44,

$$F_{cv}^{-\frac{v\beta}{n}} = q_c'', \tag{134}$$

q_c'' being a rational function of w^c. By (132), (133), (134), and other corresponding equations, the second of equations (131) becomes

$$k_c = w^r(A_{cv}A_c^{-v})Q(q_c q_c' q_c''\ldots). \tag{135}$$

In like manner, from the first of equations (131),

$$k_1 = w^a(A_v A_1^{-v})Q(q_1 q_1' q_1''\ldots),$$

q_1, q_1', etc., being what q_c, q_c', etc., become in passing from w^c to w. It may be noted that this assumes that we are entitled to change equation (133) into

$$(P_{vm}^m P_m^{-vm})^{\frac{1}{n}} = q_1'.$$

The warrant for this lies in the fact that the roots $P_m^{\frac{m}{n}}$, $P_{2m}^{\frac{m}{n}}$, $P_{3m}^{\frac{m}{n}}$, or $P_m^{\frac{1}{4}}$, $P_{2m}^{\frac{1}{4}}$, $P_{3m}^{\frac{1}{4}}$, were taken with the values they have in the root

$$P_0^{\frac{1}{4}} + P_m^{\frac{1}{4}} + P_{2m}^{\frac{1}{4}} + P_{3m}^{\frac{1}{4}}$$

of a pure uni-serial Abelian quartic. This being so, the equation

$$(P_{vm}^m P_m^{-vm})^{\frac{1}{n}} = q_1'$$

corresponds to equation (3), while (133) corresponds to (5), and, by § 5, equations

(3) and (5) subsist together. In k_1, by the Corollary in §4, we can change w into w^e. Therefore

$$k_e = w^{ac}(A_{cv}A_e^{-v})\,Q\,(q_e q_e' q_e'' \ldots).$$

By comparing this with (135), $w^r = w^{ac}$. Therefore the first of equations (127) becomes

$$R_{cv}^{\frac{1}{n}} = w^{ac}A_{cv}Q\,(P_{cvm}^{m}\phi_{cv\sigma}^{\sigma}\ \ldots)^{\frac{1}{n}}.$$

Replacing Q by $(F_{cv\beta}^{\beta}\ldots)^{\frac{1}{n}}$, and putting e for c, which we are entitled to do because w^c may be any one of the roots included in the general form w^e,

$$R_{ev}^{\frac{1}{n}} = w^{ea}A_{ev}\,(P_{evm}^{m}\phi_{ev\sigma}^{\sigma}\ \ldots\ F_{ev\beta}^{\beta})^{\frac{1}{n}},$$

which is the form of $R_{ev}^{\frac{1}{n}}$ in (110).

Sufficiency of the Forms.

§57. Here we assume that R_1 has the form (104), and that the forms in (111) are determined by the equations (109), (110), etc., while $R_0^{\frac{1}{n}}$ receives its rational value; and we have to prove that the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree, provided always that the equation of the n^{th} degree, of which it is the root, is irreducible. *In the first place*, it has been shown that there is an n^{th} root of R_0 which has a rational value; and, by hypothesis, $R_0^{\frac{1}{n}}$ has been taken with this rational value. *In the second place*, an equation of the type (3) subsists for every integral value of z. For

$$R_z R_1^{-z} = (A_z A_1^{-z})^n (P_{mz}^{m} P_m^{-mz})(\phi_{\sigma z}^{\sigma}\phi_\sigma^{-\sigma z})\ \ldots\ (F_{\beta z}^{\beta}F_\beta^{-\beta z}).$$

But P_m is of the form of the fundamental element of the root of a pure uni-serial Abelian quartic. Therefore, by §5, $P_{mz}P_m^{-z}$ is the fourth power of a rational function of the primitive fourth root of unity w^m. Therefore, because $n = 4m$, $P_{mz}^{m}P_m^{-mz}$ is the n^{th} power of a rational function of w. Also, it can be proved, exactly as in §44, that, whether z be a multiple of s or not, $\phi_{\sigma z}^{\sigma}\phi_\sigma^{-\sigma z}$ is the n^{th} power of a rational function of w. And so of the other corresponding expressions. Therefore $R_z R_1^{-z}$ is the n^{th} power of a rational function of w. *In the third place*, we have to show that an equation such as (5) subsists for every corresponding equation (3). For, let z be prime to n. It is then included in e. Also, since z and e are both prime to n, ze is included in e; and unity is included in e. But, from the manner in which the root was constructed from its fundamental element, $R_e^{\frac{1}{n}}$ is determined as in (109). Therefore

$$R_1^{\frac{1}{n}} = A_1 (P_m^m \phi_\sigma^\sigma \;\ldots\; F_\beta^\beta)^{\frac{1}{n}},$$
$$R_z^{\frac{1}{n}} = A_z (P_{zm}^m \phi_{z\sigma}^\sigma \;\ldots\; F_{z\beta}^\beta)^{\frac{1}{n}},$$
$$R_{ez}^{\frac{1}{n}} = A_{ez} (P_{ezm}^m \phi_{ez\sigma}^\sigma \;\ldots\; F_{ez\beta}^\beta)^{\frac{1}{n}}.$$

Therefore
$$\left.\begin{aligned} (R_z R_1^{-z})^{\frac{1}{n}} &= (A_z A_1^{-z}) (P_{zm} P_m^{-z})^{\frac{m}{n}} (\phi_{z\sigma} \phi_\sigma^{-z})^{\frac{\sigma}{n}} \ldots \\ (R_{ez} R_e^{-z})^{\frac{1}{n}} &= (A_{ez} A_e^{-z}) (P_{ezm} P_{em}^{-z})^{\frac{m}{n}} (\phi_{ez\sigma} \phi_{e\sigma}^{-z})^{\frac{\sigma}{n}} \ldots \end{aligned}\right\} \qquad (136)$$
and

Because $(P_{zm} P_m^{-z})^{\frac{m}{n}}$ and other such expressions have been shown to be rational functions of the primitive n^{th} root of unity, the two equations (106) correspond respectively to (3) and (5). If z be not prime to n, and yet not a multiple of n, it may be taken to be ev, where v is equal to $\frac{n}{y}$, y being one of the terms in the series (107) distinct from n, and w^e being the general primitive n^{th} root of unity. Then, just as we obtained the pair of equations (136) by means of (109), we can now, by means of (110), obtain

$$\left.\begin{aligned} (R_{ev} R_1^{-ev})^{\frac{1}{n}} &= (A_{ev} A_1^{-ev}) (P_{evm} P_m^{-ev})^{\frac{m}{n}} \ldots \\ (R_{cev} R_c^{-ev})^{\frac{1}{n}} &= (A_{cev} A_c^{-ev}) (P_{cevm} P_{cm}^{-ev})^{\frac{m}{n}} \ldots \end{aligned}\right\} \qquad (137)$$

where w^c represents any one of the primitive n^{th} roots of unity. Because $(P_{evm} P_m^{-ev})^{\frac{m}{n}}$ and other such expressions have been shown to be rational functions of the primitive n^{th} root of unity, the two equations (137) correspond respectively to (3) and (5). Finally, should z be a multiple of n, it may be taken to be zero. Then the equation corresponding to (3) is

$$R_z^{\frac{1}{n}} = q_1 R^{\frac{z}{n}},$$

q_1 being a rational function of w. Or, since $z = 0$,

$$R_0^{\frac{1}{n}} = q_1.$$

But $R_0^{\frac{1}{n}}$ is rational. Therefore q_1 is rational. Hence, if q_e be what q_1 becomes in passing from w to w^e, $q_e = q_1$. Also $R_{ez}^{\frac{1}{n}} = R_0^{\frac{1}{n}} = q_e$. Therefore, since $R_e^{\frac{z}{n}} = 1$,

$$R_{ez}^{\frac{1}{n}} = q_e R_e^{\frac{z}{n}},$$

which is the equation corresponding to (5). Therefore, whatever z be, the equation (5) subsists along with (3). Hence, by the Criterion in § 10, the expression (105) is the root of a pure uni-serial Abelian equation of the n^{th} degree.

Solvable Irreducible Equations of Prime Degrees.

§ 58. Let $f(x) = 0$ be a solvable irreducible equation of the prime degree n. Even if it be not a pure Abelian, the necessary and sufficient forms of its roots can, by means of the problems solved above, be determined *in all cases in which n is either the continued product of a number of distinct primes or four times the continued product of a number of distinct odd primes.*

§ 59. It is known that the root of the equation is of the form

$$k + R_1^{\frac{1}{n}} + R_2^{\frac{1}{n}} + \ldots . + R_{n-1}^{\frac{1}{n}}, \tag{138}$$

where k is rational; and

$$R_1, R_2, \ldots ., R_{n-1}, \tag{139}$$

are the roots of an equation of the n^{th} degree, that is, of an equation with rational coefficients. Let this equation be $\phi(x) = 0$. The root of the equation $f(x) = 0$ may also be expressed in the form

$$k + R_1^{\frac{1}{n}} + a_1 R_1^{\frac{2}{n}} + b_1 R_1^{\frac{3}{n}} + \ldots . + c_1 R_1^{\frac{n-1}{n}}, \tag{140}$$

where a_1, b_1, etc., are rational functions of R_1. The separate members of the expression (140) are severally equal to those of the expression (138); that is,

$$R_2^{\frac{1}{n}} = a_1 R_1^{\frac{2}{n}},\ R_3^{\frac{1}{n}} = b_1 R_1^{\frac{3}{n}}, \ldots ., R_{n-1}^{\frac{1}{n}} = c_1 R_1^{\frac{n-1}{n}}. \tag{141}$$

Therefore $R_2 = a_1^n R_1^2$. Hence, since a_1 is a rational function of R_1, R_2 is a rational function of R_1. The expression R_1 is thus the root of a pure Abelian equation, which, moreover, is known to be capable of having its roots arranged in a single circulating series, and therefore to be what we have called a pure uni-serial Abelian. A quotation from a remarkable memoir which was presented in 1853 by Herr Leopold Kronecker to the Academy of Berlin, and of which a translation is given in Serret's Cours d'Algèbre Supérieure (Vol. II, p. 654, 3d edition), will show how the case stands. In Kronecker's memoir μ indicates the degree of the equation, and is therefore our n, while A, B, C, etc., are quantities involved rationally in the coefficients of the equation $f(x) = 0$. Having given, after Abel, what are substantially the two forms (138) and (140), Kronecker adds: "Il est bien vrai que toute fonction algébrique, satisfaisant au problème proposé, doit pouvoir se mettre sous ces deux formes; mais ces formes sont encore trop générales, c'est-à-dire qu'elles renferment des fonctions algébriques qui ne répondent pas à la question. Je les ai donc étudiées de plus près, et j'ai trouvé d'abord que parmi les fonctions renfermées dans la forme (2)" [the

same as (138)] "celles qui satisfont au problème proposé doivent avoir la propriété nonseulment que les fonctions symétriques de R_1, R_2, etc., soient rationnelles en A, B, C, etc. (ce qu'Abel a remarqué), mais aussi que les fonctions cycliques des quantités R_1, R_2, etc., prises dans un certain ordre, soient également rationnelles en A, B, C, etc.; en d'autres termes, l'équation de degré $\mu-1$, dont R_1, R_2, etc., sont les racines, doit être une équation abélienne. J'entendrai toujours ici par équations abéliennes cette classe particulière d'équations résoluble qu'Abel a considérées dans le Memoire XI du premier volume des *Œuvres complètes*, et dont je supposerai les coefficients fonctions rationnelles de A, B, C, etc. En désignant par $x_1, x_2, \ldots, x_n$, des racines prises dans un ordre déterminé, ces équations peuvent être définies soit en disant que les fonctions cycliques des racines sont rationnelles en A, B, C, etc., soit en disant qu'on a les relations,

$$x_2 = \theta(x_1),\ x_3 = \theta(x_2),\ \ldots,\ x_n = \theta(x_{n-1}),\ x_1 = \theta x_n,$$

où $\theta(x)$ est une fonction entière de x dont les coefficients sont rationnels en A, B, C, etc." In saying that the $\mu-1$ (or, in our notation, the $n-1$) terms, R_1, R_2, etc., are the roots of an Abelian equation, Kronecker must be understood to assume that the equation $\phi(x)=0$, which has the terms in (139) for its roots, is irreducible. As a matter of fact, in the most general case, which includes all the others, the equation $\phi(x)=0$ is irreducible. But in particular cases it may be reducible, and then it is not an Abelian. In a paper by the present writer, entitled "Principles of the Solution of Equations of the Higher Degrees," which appeared in this Journal (Vol. VI, No. 1), it was proved that when the equation $\phi(x)=0$ is reducible, it can be broken into a number of irreducible equations,

$$\psi_1(x)=0,\ \psi_2(x)=0,\ \ldots,\ \psi_z(x)=0,$$

each a pure uni-serial Abelian. Hence, for a detailed discussion of the problem we have now before us, we should require to deal not only with the general case in which the equation $\phi(x)=0$ is irreducible, but also with the several cases in which equations such as $\psi_1(x)=0$, $\psi_2(x)=0$, etc., can be formed. But since, as has been stated above, the particular cases are included in the general, we shall confine ourselves to the problem of the necessary and sufficient forms of the roots of the solvable irreducible equation $f(x)=0$ of degree n, when the subordinate equation $\phi(x)=0$ of degree $n-1$ is irreducible, and is therefore a pure uni-serial Abelian; it being understood that $n-1$ is either the continued product of a number of distinct primes, or four times the continued product of a number of distinct odd primes.

Form of the Root.

§60. The solutions of the problems investigated in the preceding part of the paper have furnished us with the necessary and sufficient form of the root of the pure uni-serial Abelian equation $\phi(x) = 0$ of degree $n-1$. Let this be r_1. Let

$$w,\ w^{\lambda},\ w^{\lambda^2},\ \ldots\ldots,\ w^{\lambda^{n-2}} \tag{142}$$

be a cycle containing all the primitive n^{th} roots of unity. We may assume that λ is less than n. Let

$$1,\ \lambda,\ \alpha,\ \beta,\ \ldots\ldots,\ \delta,\ \varepsilon,\ \theta \tag{143}$$

be the indices of the powers of w in (143); that is, $\alpha = \lambda^2$, $\beta = \lambda^3$, and so on. The $n-1$ roots of the equation $\phi(x) = 0$ can be arranged in a single circulating series. Let them, so arranged, be

$$r_1,\ r_\lambda,\ r_\alpha,\ \ldots\ldots,\ r_\varepsilon,\ r_\theta. \tag{144}$$

It will be found that the terms $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., in (138), which are the same, in a certain order, as $R_1^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$, $R_\alpha^{\frac{1}{n}}$, etc., with multiples of n rejected from the subscripts, are given by the equations

$$\left.\begin{aligned} R_1^{\frac{1}{n}} &= A_1 \left(r_1^\theta r_\lambda^\varepsilon r_\alpha^\delta \ldots\ldots r_\delta^\alpha r_\varepsilon^\lambda r_\theta\right)^{\frac{1}{n}} \\ R_\lambda^{\frac{1}{n}} &= A_\lambda \left(r_1 r_\lambda^\theta r_\alpha^\varepsilon \ldots\ldots \quad r_\varepsilon^\alpha r_\theta^\lambda\right)^{\frac{1}{n}} \\ R_\alpha^{\frac{1}{n}} &= A_\alpha \left(r_1^\lambda r_\lambda r_\alpha^\theta \ldots\ldots \quad r_\theta^\alpha\right)^{\frac{1}{n}} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ R_\theta^{\frac{1}{n}} &= A_\theta \left(r_1^\varepsilon r_\lambda^\delta \quad \ldots\ldots \quad r_\theta^\theta\right)^{\frac{1}{n}} \end{aligned}\right\} \tag{145}$$

In (145) the subscripts of the factors of the expression for $R_1^{\frac{1}{n}} A_1^{-1}$ are the terms in (143), while the indices are the terms in (143) in reverse order. Because the series (144) circulates, R_λ is formed from R_1 by changing r_1 into r_λ, and, through the same change, R_λ becomes R_α, and so on.

Necessity of the above Forms.

§61. Here, assuming that the root of a solvable irreducible equation of degree n is expressible as in (138), we have to show that $R_1^{\frac{1}{n}}$, $R_2^{\frac{1}{n}}$, etc., have the forms (145).

§62. In (138) $R_1^{\frac{1}{n}}$ is an n^{th} root of R_1, one of the roots of a pure uni-serial Abelian equation $\phi(x) = 0$, the series of whose roots is contained in (139). But

R_1 may be any one of the roots. This implies that if the roots, in the order in which they circulate, are

$$R_1,\ R_\lambda,\ R_\alpha,\ \ldots\ldots,\ R_\delta,\ R_\epsilon,\ R_\theta,$$

the change of $R_1^{\frac{1}{n}}$ in the system of equations (141) into $R_\lambda^{\frac{1}{n}}$ will cause $R_\lambda^{\frac{1}{n}}$ to become $R_\alpha^{\frac{1}{n}}$, and $R_\alpha^{\frac{1}{n}}$ to become $R_\beta^{\frac{1}{n}}$, and so on. In fact, by exactly the same reasoning as that used in establishing the Criterion of pure uni-serial Abelianism, it can be made to appear that the n values of the expression (138) or of (140) obtained by taking the n values of $R_1^{\frac{1}{n}}$ for a given value of R_1, and taking at the same time the appropriate values of $R_2^{\frac{1}{n}}$, $R_3^{\frac{1}{n}}$, etc., as determined by the equations (141), would not be the roots of an equation of the n^{th} degree with rational coefficients unless $R_\lambda^{\frac{1}{n}}$ could replace $R_1^{\frac{1}{n}}$ in the manner above indicated. In like manner, by changing $R_1^{\frac{1}{n}}$ in the system of equations (141) into $R_\alpha^{\frac{1}{n}}$, $R_\lambda^{\frac{1}{n}}$ becomes $R_\beta^{\frac{1}{n}}$, and so on. The principle can be extended to all the terms in the series

$$R_1^{\frac{1}{n}},\ R_\lambda^{\frac{1}{n}},\ R_\alpha^{\frac{1}{n}},\ \ldots\ldots,\ R_\epsilon^{\frac{1}{n}},\ R_\theta^{\frac{1}{n}}. \tag{146}$$

§63. Let, then, the system of equations (141) be written

$$R_{e\lambda}^{\frac{1}{n}} = a_e' R_e^{\frac{\lambda}{n}},\ R_{e\alpha}^{\frac{1}{n}} = b_e' R_e^{\frac{\alpha}{n}},\ \text{etc.,} \tag{147}$$

e being a general symbol under which all the terms in the series (143) are contained, while a_e', b_e', etc., are rational functions of R_e. These equations give us

$$(R_e^\theta R_{e\lambda}^\epsilon R_{e\alpha}^\delta \ldots\ldots R_{e\delta}^\alpha R_{e\epsilon}^\lambda R_{e\theta})^{\frac{1}{n}} = G_e R_e^{\frac{t}{n}},$$

where G_e is a rational function of R_e, and

$$t = \theta + \epsilon\lambda + \delta\alpha + \ldots\ldots + \theta = (n-1)\,\theta = (n-1)\,\lambda^{n-2}.$$

Because λ is a prime root of n, $(n-1)\,\lambda^{n-2}$ is prime to n. Therefore t is prime to n. Therefore whole numbers h and k exist such that

$$ht = kn + 1.$$

Therefore $$(R_e^\theta R_{e\lambda}^\epsilon \ldots\ldots R_{e\theta})^{\frac{h}{n}} = (G_e^h R_e^k)\, R_e^{\frac{1}{n}}.$$

For every integral value of z, let $(R_{ez}^h)^{\frac{1}{n}}$ be written $r_{ez}^{\frac{1}{n}}$. Then, putting A_e^{-1} for $G_e^h R_e^k$,

$$R_e^{\frac{1}{n}} = A_e\,(r_e^\theta r_{e\lambda}^\epsilon r_{e\alpha}^\delta \ldots\ldots r_\delta^\alpha r_\epsilon^\lambda r_\theta)^{\frac{1}{n}}. \tag{148}$$

Because r_{es} is simply another way of writing R^h_{ez}, and the terms R_1, R_λ, etc., are the roots of a pure uni-serial Abelian, it follows that r_1, r_λ, etc., have the forms of the roots of a pure uni-serial Abelian. By putting e, then, in (148) successively equal to $1, \lambda, \alpha, \ldots\ldots, \theta$, the $n-1$ terms in (146) are obtained with the forms assigned to them in (145).

Sufficiency of the Forms.

§64. We here assume that the terms forming the series (146) are taken as in (145), and we have to show that the expression (140) is the root of a solvable irreducible equation of the n^{th} degree; provided always that the equation of the n^{th} degree, of which it is a root, is irreducible. Because the terms forming the series (146) are taken as in (145), the system of equations (147) subsists. Therefore, by a course of reasoning precisely similar to that used in an earlier part of the paper to show that the n values of the expression (2), obtained by giving s successively the values $0, 1, 2, \ldots\ldots, n-1$, are the roots of an equation of the n^{th} degree, it can now be shown that the n values of the expression (140), obtained by taking the n values of $R_1^{\frac{1}{n}}$ for a given value of R_1, are the roots of an equation of the n^{th} degree, that is, of an equation of the n^{th} degree with rational coefficients.